56. Let $u = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ k \\ 1 \end{bmatrix}$. Find the angle between u and v and all $k \in \mathbb{R}$ such that $u \perp v$.

57. Let V be a subspace of $\mathbb{R}^n$. Prove that $V^\perp$ is also a subspace of $\mathbb{R}^n$.

58. Let V and W be subspaces of $\mathbb{R}^n$. Prove that $V \perp W$ implies that $V \cap W = \{0\}$.

59. Let $P = \Big\{[a \quad b \quad c]^T \in \mathbb{R}^3 : \ a + 2b - c = 6\Big\}$.

    (a) Give three points that are in P and three points that are not in P. Is P a subspace of $\mathbb{R}^3$?

    (b) Find the subspace Q of $\mathbb{R}^3$ that has dimension 2 and no point in common with P. Give three points that are in Q and three points that are not in Q.

    (c) Find $Q^\perp$. What is the dimension of $Q^\perp$? Give three points that are in $Q^\perp$ and three points that are not in $Q^\perp$.

    (d) Find, if possible, matrices A and B such that $\mathcal{N}(A) = Q^\perp$ and $\mathcal{R}(B) = Q^\perp$.

60. Let $S = \Big\{[a \quad b \quad c \quad d]^T \in \mathbb{R}^4 : \ a + b + c + d = 0\Big\}$. Find $S^\perp$.

61. Give an example of a 2×3-matrix A with

    (a) $\mathcal{N}(A) \neq \{0\}$ and check that $\mathcal{N}(A) = \mathcal{N}(A^T A)$;

    (b) $\mathcal{N}(A) = \{0\}$ and check that $A^T A$ is invertible.

62. Let A be an $m \times n$-matrix. Prove

    (a) $\mathcal{N}(A) = \mathcal{N}(A^T A)$ (b) If $\mathcal{N}(A) = \{0\}$, then $A^T A$ is invertible.

63. Consider a company that prints books.

    (a) If no books are printed, the costs for the company are \$2000. If 100 books are printed, the costs are \$5000. For 200 and 300 books the costs are \$8000 and \$11000, respectively. Draw the data into a coordinate system, using the number of books divided by 100 on the x-axis and the costs in Dollars divided by 1000 on the y-axis. Can you find a line such that each of the points is on that line? (Will work unless you did a mistake.) Give the equation $l(x)$ of this line. Use it to estimate the costs for producing 400 books.

    (b) Same problem as in (a), but the data are now as follows: For 0, 100, 200, and 300 books the costs are \$1500, \$6200, \$8700, and \$11000, respectively. Plot the data in a coordinate system as in (a). Try to find a line through them (won't work unless you did a mistake). Use the line $l(x)$ from (a) and compute the sum $(1.5-l(0))^2+(6.2-l(1))^2+(8.7-l(2))^2+(11-l(3))^2$. What is this sum geometrically? Now consider the line $\tilde{l}(x) = 3x + 2.1$. Evaluate the corresponding sum for the line $\tilde{l}(x)$. Now find the line that has the smallest possible such sum (give this sum). Use this line to predict the cost of producing 400 books.

64. On 10 stock exchange days the cash courses x and y of the shares of two automobile companies read as follows: 420, 429, 445, 418, 431, 459, 451, 465, 449, 473 (for x) and 495, 506, 516, 475, 493, 531, 537, 554, 547, 565 (for y). Plot the data into a coordinate system. Are they on a line? (Most likely not.) Find the line that fits the data best using the method from the previous problem. Use this line to predict the course of y when $x = 455$.