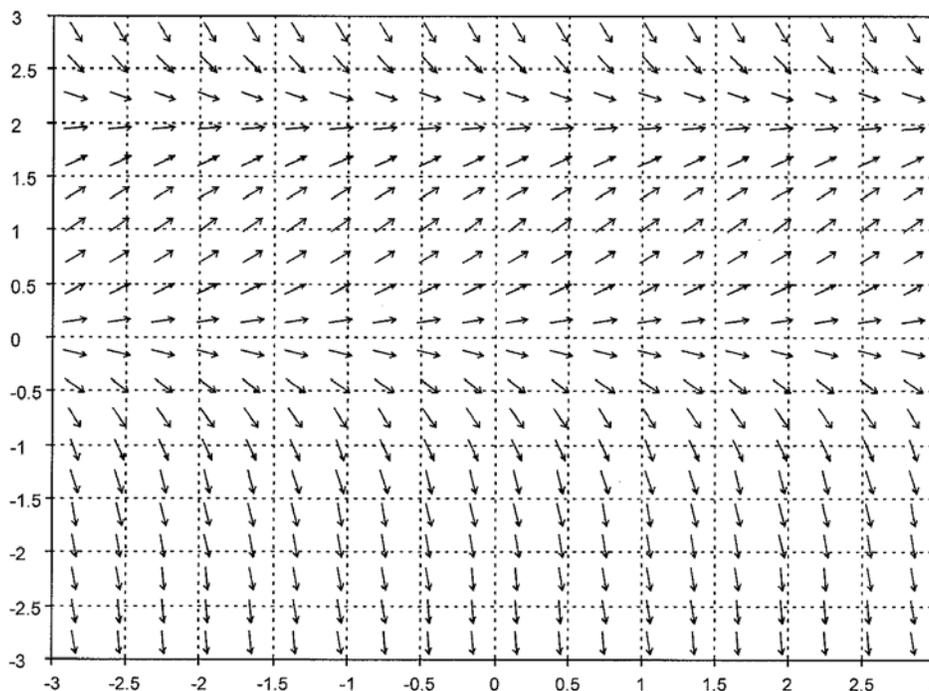**Mathematics 204   Fall 2010   Final Exam**

1.[14] Circle the letter corresponding to the differential equation whose direction field is given in the figure below.   Then explain your reasons for your answer. Note that "My calculator says so." is not a valid reason.

(a) $y' = y + 1$

(b) $y' = -y - 1$

(c) $y' = y^2 - 2y$

(d) $y' = 2y - y^2$

2.[20] Find the general solution of $y' = 2y + 4 - t$.

3.[22] (a) According to Newton's law of cooling, the temperature of an object changes at a rate proportional to the difference between the temperature of the object and the ambient temperature (i.e. the temperature of the surroundings). Express Newton's law of cooling as a differential equation.

(b) A murder victim was found on the street at 12:00 midnight and the temperature of the body at that time was 85°F. After 2 hours, the temperature of the body had dropped to 74°F. Between 9:00 PM and 2:00 AM, the outdoor temperature was steady at 68°F. Use Newton's law of cooling to determine the time the crime took place. You should assume the temperature of the body at the time of death was a standard 98.6°F, and be sure to give your answer in terms of a time of day as read on a clock - for example, 9:38 PM.

4.[20] (a) Solve $y''' - y = 0$. You may find the identity $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ useful.

(b) Solve $t^2 y'' + 3ty' + y = 0$ on the interval $t > 0$.

5.[22] Consider the differential equation $y'' - 2y' + y = \dfrac{e^t}{1+t^2}$.

(a) Classify the differential equation by giving its order, stating whether it is linear or nonlinear, homogeneous or nonhomogeneous, and whether it has constant or variable coefficients.

(b) Find the general solution of the differential equation.

6.[15] (Please use 32 feet per second per second as the acceleration of gravity in this problem.) A spring hangs vertically from a rigid support. A body weighing 16 pounds stretches the spring 4 inches when it is first attached. The body is in a medium with a damping constant of 2 pound-seconds per foot. Suppose the body is displaced 3 inches above the equilibrium position and then released. If gravity is the only external force that acts on the body, set up, **BUT DO NOT SOLVE**, an initial value problem describing the body's motion.

7.[21] Use the Laplace transform to solve the integral equation $y(t) + 3\displaystyle\int_0^t e^{-4(t-\tau)} y(\tau)\,d\tau = 1$.

8.[22] Solve the initial value problem $y'' + 4y = f(t)$, $y(0) = 1$, $y'(0) = 0$, if $f(t) = \begin{cases} 0 & \text{if } t < 2, \\ -4 & \text{if } t \geq 2. \end{cases}$

Then compute the value of the solution, accurate to three decimal places, when $t = 1$ and $t = 3$.

9.[22] Find the general solution of the system $\mathbf{x}' = \begin{pmatrix} -6 & 4 \\ -1 & -2 \end{pmatrix} \mathbf{x}$ and describe the behavior of solutions as $t \to \infty$.

10.[22] Given that $\mathbf{x}(t) = c_1 \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} 2e^{-2t} \\ 3e^{-2t} \end{pmatrix}$ is the general solution for the homogeneous system

$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}$, solve the nonhomogeneous initial value problem $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^{-t}$, $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

1.[12] State the order of each of the following differential equations. Is the equation linear or nonlinear? If the equation is linear, is it homogeneous?

	Order?	Linear?	Homogeneous?
$y''' + ty' + \cos^3(t)\,y - t^3 = 0$			
$x'' + t\ln(x)\,x = 0$			
$\dfrac{d^3 y}{dx^3} + \left(\dfrac{dy}{dx}\right)^4 + y = 0$			
$y'' + 10y - \delta(t-2) = 0$			
$\dfrac{dp}{dt} + p(p-1) = 0$			

2.[20] Find an explicit solution of the initial value problem $y' = \dfrac{x}{y}$, $y(0) = -1$.

3.[20] Solve the differential equation $x^2 y' = 1 - xy$ when $x > 0$.

4.[10] According to Newton's empirical law of cooling/warming, the rate of change of the temperature $T(t)$ of a body at time t is proportional to the difference between the temperature of the body and the temperature T_m of the surrounding medium. State this law in the form of a differential equation. Please **DO NOT SOLVE** your differential equation.

5.[18] A tank initially contains 120 liters of pure water. A solution containing 1.5 grams of salt per liter enters the tank at a rate of 2 liters per minute, and the well-stirred mixture leaves the tank at the same rate. Set up, **BUT DO NOT SOLVE**, an initial value problem that models the amount of salt in the tank at any time $t \geq 0$.

6.[21] Find the general solution of $y^{(4)} - 16y = e^x$. You may find the following identity useful:
$$a^4 - b^4 = (a-b)(a+b)(a^2 + b^2).$$

7.[20] Solve the initial value problem $y'' + y = \delta(t - \pi)$, $y(0) = 0$, $y'(0) = 1$.

8.[15] A spring hangs vertically from a rigid support. When a 4 pound object is attached to the end of the spring, the object stretches the spring $1/2$ foot. Suppose the object is released from rest from a point 1 foot above the equilibrium position. Assuming air resistance (or the damping force) at any instant is equal to one-half the instantaneous velocity of the object, write, **BUT DO NOT SOLVE**, an initial value problem that models the motion of the object if it is driven by an external downward force at time t of $\sin(2t)$ pounds.

9.[20] Use the Laplace transform method to solve the initial value problem
$$y'' - 4y' + 4y = t^3 e^{2t}, \quad y(0) = y'(0) = 0.$$
NO CREDIT will be awarded for any other method of solution.

10.[21] Solve the initial value problem $\mathbf{X}' = \begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix} \mathbf{X}$, $\mathbf{X}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Bonus [1]: Sketch this trajectory of the system in the phase plane.

11.[21] Use the method of variation of parameters for systems to solve the system
$$\frac{dx}{dt} = -y + \sec(t)$$
$$\frac{dy}{dt} = x$$
given that the general solution to the associated homogeneous system is
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} + c_2 \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}.$$