31. For a sequence y_0, y_1, y_2, $\ldots$, we put $\Delta y_k = y_{k+1} - y_k$ and then recursively $\Delta^{n+1} y_k = \Delta(\Delta^n y_k)$.
   (a) Find Δy_k if $y_k = 1$, if $y_k = k$, if $y_k = k^2$, and if $y_k = 2^k$. Also find $\Delta^2 y_k$ in all of these cases.
   (b) Find $\Delta^2 y_k$ and $\Delta^3 y_k$ for arbitrary y_k.
   (c) Prove the discrete product rule: $\Delta(y_k x_k) = y_k \Delta x_k + x_{k+1} \Delta y_k$. Find the quotient rule.
   (d) Solve $\Delta y_k = 2y_k$. Also, solve $\Delta y_k = \rho y_k$, $y_0 = 1$.
32. Recall from calculus that $y'(t) = \lim_{h\to 0} \frac{y(t+h)-y(t)}{h}$ and consider the IVP $y' = y$, $y(0) = 1$.
   (a) First of all, plot the solution of the above problem for $0 \le t \le 1$.
   (b) Now let $h = 1$, $t = 0$ in the above formula, replace $y'(0)$ by the obtained expression, put $y_1 = y(1)$ and $y_0 = y(0)$, replace the $y(0)$ in the initial value problem by y_0, write down the corresponding difference equation, solve it, and plot y_0 and y_1 in a coordinate system.
   (c) Now let $h = 0.5$, $y_0 = y(0)$, $y_1 = y(0.5)$, $y_2 = y(1)$ and do the same things as in (b).
   (d) Do all of the above for $h = 0.25$ and for $h = 0.1$.
   (e) If you like, use a computer to do all of the above for $h = 0.01$ and for $h = 0.00001$. In any case, explain what will happen as h is tending to 0.
33. Given are three pegs standing next to each other. On the first peg there are n rings of different size, the largest one on the bottom, then the second largest one, and so on, the smallest one on the top. The problem is to find the minimum number of moves (denote this number by y_n) required to move all n rings from the first peg to the third peg. A move consists of transferring a single ring from one peg to another with the restriction that a larger ring may not be placed on a smaller ring.
   (a) Find y_1, y_2, y_3, and y_4. Sketch pictures of each starting position, intermediate positions, and final position. There is one intermediate position that will occur in any case. Which one? Sketch this intermediate position, together with the starting and the final positions for $n = 5$.
   (b) Come up with a difference equation for y_n.
   (c) Solve it to obtain y_n for any natural number n.
34. If $b^2 - 4ac > 0$, solve the IVP $ay'' + by' + cy = 0$, $y(t_0) = y_0$, $y'(t_0) = y_0'$.
35. Solve the following initial value problems:
   (a) $y'' - 3y' - 10y = 0$. First, $y(0) = 1$, $y'(0) = 0$. Next, $y(0) = 0$, $y'(0) = 1$;
   (b) $6y'' - 5y' + y = 0$. First, $y(0) = 4$, $y'(0) = 0$. Next: $y(0) = 0$, $y'(0) = 0$;
   (c) $y'' + 3y' = 0$, $y(0) = -2$, $y'(0) = 3$;
   (d) $6y'' - 7y' + 2y = 0$, $y(0) = 0$, $y'(0) = 1$;
   (e) $2y'' - 3y' + y = 0$, $y(0) = 2$, $y'(0) = \frac{1}{2}$.
36. Consider the equation $y'' = y$.
   (a) Sketch the solutions c with $y(0) = 1$ and $y'(0) = 0$ and s with $y(0) = 0$ and $y'(0) = 1$.
   (b) Show that $c^2(t) - s^2(t) = 1$ for all t. Also, prove that $c' = s$ and $s' = c$.
   (c) Draw the Gateway Arch $y(x) = -127.7c(\frac{x}{127.7}) + 757.7$. How high is it? How long is it's base?
37. For complex z, define $\mathrm{e}^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$, $\sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$, $\cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$.
   (a) Find the first four terms in each of the above series.
   (b) Plug $\mathrm{i}w$ in place of z in the first formula and show $\mathrm{e}^{\mathrm{i}w} = \cos w + \mathrm{i} \sin w$.
   (c) Calculate $\mathrm{e}^{\pi \mathrm{i}}$ and $\mathrm{e}^{2\pi \mathrm{i}}$.
   (d) Prove the formulae $\frac{\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}}{2} = \cos z$ and $\frac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}} = \sin z$.