18. Work on problems 67–82 of Section 2.2 in the textbook.

19. Show that for two sequences $a_k$ and $b_k$ we have

(a) $\Delta (a_k + b_k) = \Delta a_k + \Delta b_k$;
(b) $\Delta (a_k - b_k) = \Delta a_k - \Delta b_k$;
(c) $\Delta (a_k b_k) = (\Delta a_k)b_k + (\Delta b_k)a_k + 1$;
(d) $\Delta \left( \frac{a_k}{b_k} \right) = \frac{(\Delta a_k)b_k - (\Delta b_k)a_k}{b_k b_{k+1}}$.

20. Show (by examining monotonicity and boundedness) that the following sequences are convergent:

(a) $a_n = \frac{1}{\sqrt{n}}$;
(b) $a_n = \frac{1}{n+5}$;
(c) $a_n = \frac{2}{\sqrt{n}} + \frac{3}{n} + 4$;
(d) $a_n = \frac{2n-6}{3n+1}$;
(e) $a_n = \frac{5n+1}{2n-3}$.

21. Let $c > 0$, $a_0 \in (0, \frac{1}{c})$, and $a_{n+1} = a_n(2 - ca_n)$ for $n \in \mathbb{N}$.

(a) For $c = 3$ and your choice of $a_0$, compute $a_k$ for $k \in \{1, 2, 3, 4, 5, 6\}$.
(b) Show that $\{a_n\}$ converges and compute its limit.

22. Let $c \geq 1$, $a_0 \geq \sqrt{c}$, and $a_{n+1} = \frac{1}{2} \left( a_n + \frac{c}{a_n} \right)$ for $n \in \mathbb{N}$.

(a) For $c = 3$ and your choice of $a_0$, compute $a_k$ for $k \in \{1, 2, 3, 4, 5, 6\}$.
(b) Show that $\{a_n\}$ converges and compute its limit.

23. Let $f_1 = 1$, $f_2 = 2$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

(a) Compute the first 10 elements of (the Fibonacci Sequence) $f_n$.
(b) Show that $f_n^2 = f_{n-1}f_{n+1} + (-1)^n$ holds for all $n \in \mathbb{N} \setminus \{1\}$.
(c) Show that $\sum_{k=1}^{n} f_k = f_{n+2} - 2$ and $\sum_{k=1}^{n} f_k^2 = f_n f_{n+1} - 1$ hold for all $n \in \mathbb{N}$.
(d) Show that $f_{n+2} - f_{n+1} = f_n f_{n+3}$ holds for all $n \in \mathbb{N}$.
(e) Show $f_n > \left( \frac{3}{2} \right)^n$ for all $n \in \mathbb{N} \setminus \{1, 2, 3, 4\}$ and $f_n < 2^n$ for all $n \in \mathbb{N}$.
(f) Prove that $\sum_{k=1}^{n} f_{2k-1} = f_{2n} - 1$ and $\sum_{k=1}^{n} f_{2k} = f_{2n+1} - 1$ hold for all $n \in \mathbb{N}$.
(g) Show that $f_n = \frac{f_{n-1} + \sqrt{5f_{n-1}^2 + 4(-1)^n}}{2}$ holds for all $n \in \mathbb{N} \setminus \{1\}$.

24. (Extra Credit 30 points) For any two numbers $a$ and $b$, use induction to prove the binomial theorem

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a + b)^n$$

for each $n \in \mathbb{N}_0$. Then consider the sequence $a_n$ defined by

$$a_n = \left( 1 + \frac{1}{n} \right)^n.$$

Use the binomial theorem to calculate $\Delta a_n$ and determine whether $a_n$ is monotone. Prove that $a_n$ is bounded. Is $a_n$ convergent?