18. Work on problems 67–82 of Section 2.2 in the textbook.
19. Show that for two sequences a_k and b_k we have
    (a) $\Delta(a_k + b_k) = \Delta a_k + \Delta b_k$;
    (b) $\Delta(a_k - b_k) = \Delta a_k - \Delta b_k$;
    (c) $\Delta(a_k b_k) = (\Delta a_k)b_k + (\Delta b_k)a_{k+1}$;
    (d) $\Delta\left(\frac{a_k}{b_k}\right) = \frac{(\Delta a_k)b_k - (\Delta b_k)a_k}{b_k b_{k+1}}$.
20. Show (by examining monotonicity and boundedness) that the following sequences are convergent:
    (a) $a_n = \frac{1}{\sqrt{n}}$;
    (b) $a_n = \frac{1}{n+5}$;
    (c) $a_n = \frac{2}{\sqrt{n}} + \frac{3}{n} + 4$;
    (d) $a_n = \frac{2n-6}{3n+1}$;
    (e) $a_n = \frac{5n+1}{2n-3}$.
21. Let $c > 0$, $a_0 \in \left(0, \frac{1}{c}\right)$, and $a_{n+1} = a_n(2 - ca_n)$ for $n \in \mathbb{N}$.
    (a) For $c = 3$ and your choice of a_0, compute a_k for $k \in \{1, 2, 3, 4, 5, 6\}$.
    (b) Show that $\{a_n\}$ converges and compute its limit.
22. Let $c \geq 1$, $a_0 \geq \sqrt{c}$, and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right)$ for $n \in \mathbb{N}$.
    (a) For $c = 3$ and your choice of a_0, compute a_k for $k \in \{1, 2, 3, 4, 5, 6\}$.
    (b) Show that $\{a_n\}$ converges and compute its limit.
23. Let $f_1 = 1$, $f_2 = 2$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.
    (a) Compute the first 10 elements of (the Fibonacci Sequence) f_n.
    (b) Show that $f_n^2 = f_{n-1}f_{n+1} + (-1)^n$ holds for all $n \in \mathbb{N} \setminus \{1\}$.
    (c) Show that $\sum_{k=1}^{n} f_k = f_{n+2} - 2$ and $\sum_{k=1}^{n} f_k^2 = f_n f_{n+1} - 1$ hold for all $n \in \mathbb{N}$.
    (d) Show that $f_{n+2}^2 - f_{n+1}^2 = f_n f_{n+3}$ holds for all $n \in \mathbb{N}$.
    (e) Show $f_n > \left(\frac{3}{2}\right)^n$ for all $n \in \mathbb{N} \setminus \{1, 2, 3, 4\}$ and $f_n < 2^n$ for all $n \in \mathbb{N}$.
    (f) Prove that $\sum_{k=1}^{n} f_{2k-1} = f_{2n} - 1$ and $\sum_{k=1}^{n} f_{2k} = f_{2n+1} - 1$ hold for all $n \in \mathbb{N}$.
    (g) Show that $f_n = \frac{f_{n-1} + \sqrt{5f_{n-1}^2 + 4(-1)^n}}{2}$ holds for all $n \in \mathbb{N} \setminus \{1\}$.
24. **(Extra Credit 30 points)** For any two numbers a and b, use induction to prove the binomial theorem

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a + b)^n$$

for each $n \in \mathbb{N}_0$. Then consider the sequence a_n defined by

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

Use the binomial theorem to calculate Δa_n and determine whether a_n is monotone. Prove that a_n is bounded. Is a_n convergent?