Part A: Fill in only the boxes and do your work on a separate sheet.

1. Please complete the following truth tables:

p	q	$p \vee q$	$\overline{p} \vee q$	$(p \vee q) \wedge \overline{\overline{p} \vee q}$
T	F	**T**	**F**	**T**
F	F	**F**	**T**	**F**
F	T	**T**	**T**	**F**
T	T	**T**	**T**	**F**

p	q	$p \wedge q$	$\overline{q}$	$(p \wedge q) \vee \overline{q}$
T	F	**F**	**T**	**T**
F	F	**F**	**T**	**T**
F	T	**F**	**F**	**F**
T	T	**T**	**F**	**T**

2. Write "T" for "true" or "F" for "false" in the boxes:

   (a) $\forall x \in \mathbb{R}\ x^2 - 9 = 0$ is **F**

   (b) $\exists x \in \mathbb{R}\ x^2 - 9 = 0$ is **T**

   (c) $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ xy = 0$ is **T**

   (d) $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ x = y^2$ is **F**

   (e) $\exists x \in \mathbb{R}\ x^4 = -1$ is **F**

   (f) $\forall x \in \mathbb{R}\ x^4 > 0$ is **F**

Part B: For the remaining problems, show your work clearly, explaining each step of the proofs. Use only the space allocated for each problem (use separate sheets of paper for additional work).

3. Prove or disprove $\exists n \in \mathbb{N}\ 3|(n^2-2)$

This is false. We show that the opposite

$$\boxed{\forall n \in \mathbb{N}\ 3 \not|(n^2-2)} \tag{1}$$

is true. Let $n \in \mathbb{N}$. Then n is of the form $3k$ for some $k \in \mathbb{N}$ or it is of the form $3k+1$ for some $k \in \mathbb{N}_0$ or it is of the form $3k+2$ for some $k \in \mathbb{N}_0$. We will show that in either case n^2-2 is not divisible by 3.

(a) **If $n = 3k$ for some $k \in \mathbb{N}$, then**

$$n^2 - 2 = 9k^2 - 2 = 3 \cdot (3k^2 - 1) + 1$$

is not divisible by 3.

(b) **If $n = 3k+1$ for some $k \in \mathbb{N}_0$, then**

$$n^2 - 2 = 9k^2 + 6k + 1 - 2 = 9k^2 + 6k - 1 = 3 \cdot (3k^2 + 2k - 1) + 2$$

is not divisible by 3.

(c) **If $n = 3k+2$ for some $k \in \mathbb{N}_0$, then**

$$n^2 - 2 = 9k^2 + 12k + 4 - 2 = 9k^2 + 12k + 2 = 3 \cdot (3k^2 + 4k) + 2$$

is not divisible by 3.

So, no matter what $n \in \mathbb{N}$ is, n^2-2 is not divisible by 3 and hence (1) is true and so the original proposition is false.

4. Prove the following statements using the Principle of Mathematical Induction:

(a) $\forall n \in \mathbb{N} \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

Let $P(n)$ be the propositional function

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 .$$

We show that

$$\boxed{\forall n \in \mathbb{N} \quad P(n)} \tag{2}$$

is true.

<u>First Step:</u> Since $1 = \left(\frac{1\cdot 2}{2}\right)^2$, we find that

$$\boxed{P(1)} \tag{3}$$

is true.

<u>Second Step:</u> Let $n \in \mathbb{N}$. If $P(n)$ is false, then $P(n) \to P(n+1)$ is true. Now assume that $P(n)$ is true. Then

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 .$$

Hence

$$\begin{aligned}
\sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^{n} k^3 + (n+1)^3 \\
&= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\
&= \frac{(n+1)^2(n^2 + 4(n+1))}{4} \\
&= \frac{(n+1)^2(n^2 + 4n + 4)}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4} \\
&= \left(\frac{(n+1)(n+2)}{2}\right)^2
\end{aligned}$$

so that $P(n+1)$ is true, and now $P(n) \to P(n+1)$ is true. Altogether, we now find that

$$\boxed{\forall n \in \mathbb{N} \quad P(n) \to P(n+1)} \tag{4}$$

is true.

<u>Conclusion:</u> Since the statements (3) and (4) are both true, we use the *Principle of Mathematical Induction* to conclude that (2) is true.

(b) $\forall n \in \mathbb{N}\ 3^n \geq n2^n$

Let $Q(n)$ be the propositional function

$$3^n \geq n2^n.$$

We show that

$$\boxed{\forall n \in \mathbb{N} \quad Q(n)} \tag{5}$$

is true.

<u>First Step:</u> Since $3^2 = 9 > 8 = 2 \cdot 2^2$, we find that

$$\boxed{Q(2)} \tag{6}$$

is true.

<u>Second Step:</u> Let $n \in \mathbb{N}$ with $n \geq 2$. Add $2n$ on both sides and note that then

$$3n \geq 2n + 2$$

holds. If $Q(n)$ is false, then $Q(n) \to Q(n+1)$ is true. Now assume that $Q(n)$ is true. Then

$$3^n \geq n2^n.$$

Hence

$$\begin{aligned} 3^{n+1} &= 3 \cdot 3^n \\ &\geq 3n2^n \\ &\geq (2n+2)2^n \\ &= 2(n+1)2^n \\ &= (n+1)2^{n+1} \end{aligned}$$

so that $Q(n+1)$ is true, and now $Q(n) \to Q(n+1)$ is true. Altogether, we now find that

$$\boxed{\forall n \in \mathbb{N} \setminus \{1\} \quad Q(n) \to Q(n+1)} \tag{7}$$

is true.

<u>Conclusion:</u> Since the statements (6) and (7) are both true, we use the *Principle of Mathematical Induction* to conclude that

$$\forall n \in \mathbb{N} \setminus \{1\}\ Q(n)$$

is true. But since $Q(1)$ is clearly also true as $3 > 2$, we find that (5) is true.