12. Prove the properties of determinants (Theorem 1.12 from the lecture notes) for arbitrary 2×2-matrices.

13. A police boat cruising for drug dealers in the Mediterranean uses a linear transformation to encrypt its position $\binom{x_N}{x_E}$ (northern latitude x_N and eastern longitude x_E) and radios the encrypted position to the headquarters in Marseille. The headquarters use another linear transformation and radio their encryption to Paris. Spies find out that, when the boat was at $\binom{42}{6}$, $\binom{54}{156}$ arrived in Marseille and $\binom{576}{258}$ arrived in Paris. Also, they know that, when the boat was at $\binom{39}{15}$, $\binom{516}{183}$ arrived in Paris. As a final piece of information they know that Marseille received a message $\binom{54}{96}$ that was sent to Paris as $\binom{396}{138}$. Now, spies catch the message $\binom{446}{113}$ in Paris. Where is the boat? Which message arrived in Marseille?

14. Let x and y be two different solutions of the same linear system $Ax = b$. Show that $\lambda x + (1 - \lambda)y$ is also a solution of the same linear system, where λ can be any real number between 0 and 1. Also explain what that means geometrically, i.e., draw a picture for the case $n = 2$.

15. Find the characteristic polynomials and all eigenvalues and corresponding eigenvectors for the following matrices A. Also find the trace and the determinant of the matrices. Then verify that the trace is the sum and the determinant is the product of the eigenvalues.

(a) $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$; (b) $A = \begin{pmatrix} -3 & 1 \\ -7 & 5 \end{pmatrix}$; (c) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$;

(d) $A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$; (e) $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$; (f) $A = \begin{pmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$;

(g) $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$; (h) $A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}$; (i) $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$.

16. For the invertible matrices of the previous problem, do the following: Calculate the inverse and its eigenvalues. Guess a connection between the eigenvalues of an invertible A and the eigenvalues of A^{-1} and prove it.