Exam #1, Math 2201, Dr. M. Bohner | Name: **Solution Key** | Feb 14, 2002.

Part A: Fill in **only** the boxes and do your work on a separate sheet.

1. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$.

   (a) $2A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

   (b) $B + B^T = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix}$

   (c) $AC = \begin{pmatrix} 3 & 7 & 11 \\ 3 & 7 & 11 \end{pmatrix}$

   (d) $ACB = \begin{pmatrix} 50 & 7 & 11 \\ 50 & 7 & 11 \end{pmatrix}$

   (e) $B^2 = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 6 & 0 & 1 \end{pmatrix}$

2. The solution of the system $u + 2v + 3w = 39$, $u + 3v + 2w = 34$, $3u + 2v + w = 26$ is

   $u = \mathbf{2.75}$, $v = \mathbf{4.25}$, $w = \mathbf{9.25}$.

3. Let $A = \begin{pmatrix} -3 & 1 \\ -7 & 5 \end{pmatrix}$.

   (a) $\det A = \mathbf{-8}$

   (b) $\operatorname{tr} A = \mathbf{2}$

   (c) $A^{-1} = \mathbf{\frac{1}{8}} \begin{pmatrix} -5 & 1 \\ -7 & 3 \end{pmatrix}$

   (d) The eigenvalues of A are $\mathbf{-2}$ **and** $\mathbf{4}$

   (e) The eigenvectors of A are $\begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{1} \\ \mathbf{7} \end{pmatrix}$

   (f) The eigenvalues of A^{-1} are $\mathbf{-1/2}$ **and** $\mathbf{1/4}$

   (g) The eigenvalues of $2A^2 + A - 3I$ are **3 and 33**

Part B: For the remaining problems, show your work clearly, explaining each step. Use only the space allocated for each problem (use separate sheets of paper for additional work).

4. Prove the formula $\det A^{-1} = \frac{1}{\det A}$ for arbitrary invertible 2×2-matrices A.

Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ **is invertible, i.e.,** $\det A = ad - bc \neq 0$**. Then**

$$A^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

so that

$$\det A^{-1} = \frac{ad}{(ad-bc)^2} - \frac{bc}{(ad-bc)^2} = \frac{ad-bc}{(ad-bc)^2} = \frac{1}{ad-bc} = \frac{1}{\det A}.$$

5. Let x and y be two different solutions of the same linear system $Ax = b$. Show that $\lambda x + (1-\lambda)y$ is also a solution of the same linear system, where λ can be any real number between 0 and 1.

Since $Ax = b$ **and** $Ay = b$**, we find that for any** $\lambda \in [0,1]$

$$\begin{aligned} A(\lambda x + (1-\lambda)y) &= A\lambda x + A(1-\lambda)y \\ &= \lambda Ax + (1-\lambda)Ay \\ &= \lambda b + (1-\lambda)b \\ &= \lambda b + b - \lambda b \\ &= b, \end{aligned}$$

so $\lambda x + (1-\lambda)y$ **also solves the same linear system.**

6. Find all eigenvalues and eigenvectors of $A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}$.

We first find the characteristic polynomial as

$$\begin{aligned} \det(A - \lambda I) &= (4-\lambda)(5-\lambda)(2-\lambda) - 2 - 2 + (5-\lambda) + 2(4-\lambda) - 2(2-\lambda) \\ &= (4-\lambda)(10 - 7\lambda + \lambda^2) - 4 + 5 - \lambda + 8 - 2\lambda - 4 + 2\lambda \\ &= 40 - 10\lambda - 28\lambda + 7\lambda^2 + 4\lambda^2 - \lambda^3 + 1 - \lambda + 8 - 2\lambda - 4 + 2\lambda \\ &= -\lambda^3 + 11\lambda^2 - 39\lambda + 45 \\ &= -(\lambda - 3)^2(\lambda - 5), \end{aligned}$$

so the eigenvalues are 3 **and** 5**. Now**

$$A - 3I = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

after Gaussian elimination, so that two linearly independent eigenvectors corresponding to the eigenvalue 3 **are given by**

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Next,

$$A - 5I = \begin{pmatrix} -1 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & 1 & -3 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & -4 \\ 0 & 2 & -4 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}$$

after Gaussian elimination, so that one linearly independent eigenvector corresponding to the eigenvalue 5 **is given by**

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

7. Suppose $r_0 = r_1 = 1$ and $r_{n+1} = r_n + 2r_{n-1}$ for each $n \in \mathbb{N}$. Use matrix powers to find a formula for r_n for each $n \in \mathbb{N}$.

Denote $u_n = r_{n+1}$ and $x = \binom{r}{u}$. Then

$$x_{n+1} = \begin{pmatrix} r_{n+1} \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} u_n \\ u_n + 2r_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} r_n \\ u_n \end{pmatrix} = Ax_n$$

with $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$. The characteristic equation of A is

$$\det(A - \lambda I) = -\lambda(1 - \lambda) - 2 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1),$$

so that A has eigenvalues -1 and 2. Note that

$$A + I = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad A - 2I = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}$$

to find that A has two linearly independent eigenvectors $\binom{1}{-1}$ and $\binom{1}{2}$. Put $P = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$. Then

$$\begin{aligned} A^n &= P \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}^n P^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2(-1)^n & -(-1)^n \\ 2^n & 2^n \end{pmatrix} = \begin{pmatrix} \frac{2(-1)^n + 2^n}{3} & \frac{2^n - (-1)^n}{3} \\ \frac{2^{n+1} - 2(-1)^n}{3} & \frac{(-1)^n + 2^{n+1}}{3} \end{pmatrix} \end{aligned}$$

and hence

$$x_n = A^n x_0 = A^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2^{n+1} + (-1)^n}{3} \\ \frac{2^{n+2} + (-1)^{n+1}}{3} \end{pmatrix}$$

so that we get

$$r_n = \frac{2^{n+1} + (-1)^n}{3}.$$