Exam #2, Math 2201, Dr. M. Bohner | Name: **Solution Key** | Mar 28, 2002.

Part A: Fill in **only** the boxes and do your work on a separate sheet.

1. Fill in the boxes:

equation	order?	linear?
$y' = t\sqrt{y}$	**first**	**no**
$y'' + 16y = 0$	**second**	**yes**
$y' = 25 + y^2$	**first**	**no**
$t^2y'' - ty' + 2y = 0$	**second**	**yes**

2. Let $N(t)$ be the number of individuals in a certain population at time t. If we assume that N increases proportionally to the number of individuals currently present, give a differential equation for N: $\boxed{\mathbf{N' = cN}}$. For an initial condition, assume that at time 0 the number of individuals is N_0. The solution is $\boxed{N(t) = \mathbf{N_0 e^{ct}}}$. If $\boxed{N_0 > \mathbf{0}}$, then the solution is increasing with $\boxed{\lim_{t\to\infty} N(t) = \infty}$; If $\boxed{N_0 < \mathbf{0}}$, then the solution is decreasing with $\boxed{\lim_{t\to\infty} N(t) = -\infty}$; If $\boxed{N_0 = \mathbf{0}}$, then the solution is constant with $\boxed{\lim_{t\to\infty} N(t) = \mathbf{0}}$. Now we assume that N changes proportionally to the product of the number of individuals currently present and $(1 - \frac{N(t)}{K})$, where K is a constant. The corresponding ODE is $\boxed{\mathbf{N' = cN\left(1 - \frac{N}{K}\right)}}$. It can be solved using the $\boxed{\textbf{separation of variables} \text{ technique}}$. There are two constant solutions: $\boxed{N_1(t) \equiv \mathbf{0}}$ and $\boxed{N_2(t) \equiv \mathbf{K}}$.

3. Fill in the boxes (y_1 and y_2 should be linearly independent):

equation	characteristic equation	zeros	$y_1(t)$	$y_2(t)$
$y'' - 3y' - 10y = 0$	$\mathbf{r^2 - 3r - 10 = 0}$	**5 and -2**	$\mathbf{e^{5t}}$	$\mathbf{e^{-2t}}$
$y'' + 3y' = 0$	$\mathbf{r^2 + 3r = 0}$	**0 and -3**	**1**	$\mathbf{e^{-3t}}$
$y'' + 4y = 0$	$\mathbf{r^2 + 4 = 0}$	**2i and -2i**	$\mathbf{\sin(2t)}$	$\mathbf{\cos(2t)}$
$y'' + 4y' + 5y = 0$	$\mathbf{r^2 + 4r + 5 = 0}$	**-2+i and -2-i**	$\mathbf{e^{-2t}\cos(t)}$	$\mathbf{e^{-2t}\sin(t)}$
$y'' - 4y' + 4y = 0$	$\mathbf{r^2 - 4r + 4 = 0}$	**2**	$\mathbf{e^{2t}}$	$\mathbf{te^{2t}}$

Part B: For the remaining problems, show your work clearly, explaining each step. Use only the space allocated for each problem (use separate sheets of paper for additional work).

4. Solve the problem $y' = \frac{t^2}{y}$, $y(0) = -1$ by separating the variables. Give the solution explicitly.

Suppose y solves the problem. Then

$$y(t)y'(t) = t^2 \quad \textbf{with} \quad y(0) = -1.$$

We integrate both sides of the equation between 0 and t to obtain

$$\int_0^t y(s)y'(s)ds = \int_0^t s^2 ds,$$

i.e.,

$$\frac{y^2(t)}{2} - \frac{y^2(0)}{2} = \frac{t^3}{3}$$

and therefore (note $y(0) = -1$)

$$y(t) = -\sqrt{1 + \frac{2t^3}{3}}.$$

It is also easy to check that this y indeed solves the problem.

5. Solve the problem $y' + 2y = te^{-2t}$, $y(1) = 0$.

Suppose y solves the problem. Then

$$y'(t) + 2y(t) = te^{-2t} \quad \textbf{with} \quad y(1) = 0.$$

We multiply both sides of the equation with e^{2t} to obtain

$$\frac{d}{dt}\left[y(t)e^{2t}\right] = t.$$

Now integrate both sides of the equation between 1 and t:

$$y(t)e^{2t} - y(1)e^2 = \frac{t^2}{2} - \frac{1}{2}$$

so that (note $y(1) = 0$)

$$y(t) = \frac{t^2 - 1}{2}e^{-2t}.$$

It is also easy to check that this y indeed solves the problem.

6. A tank has ten gallons of water in which two pounds of salt has been dissolved. Brine with 1.5 pound of salt per gallon enters at three gallons per minute, and the well-stirred mixture is drained out at a rate of four gallons per minute. Find the amount of salt in the tank at any time.

Rate in is

$$1.5\frac{\text{lb}}{\text{gal}} \cdot 3\frac{\text{gal}}{\text{min}} = 4.5\frac{\text{lb}}{\text{min}}$$

and rate out is

$$\frac{A(t)\ \text{lb}}{(10-t)\ \text{gal}} \cdot 4\frac{\text{gal}}{\text{min}} = \frac{4A(t)}{(10-t)}\frac{\text{lb}}{\text{min}}.$$

Therefore we need to solve the equation

$$A' = 4.5 - \frac{4A}{10-t}.$$

Suppose A **is a solution of this equation so that**

$$4.5 = A'(t) + \frac{4A(t)}{10-t}.$$

We multiply both sides of this equation with

$$\exp\left[\int\left(\frac{4}{10-t}dt\right)\right] = \frac{1}{(10-t)^4}$$

to obtain

$$\frac{d}{dt}\left[\frac{1.5}{(10-t)^3}\right] = \frac{4.5}{(10-t)^4} = \frac{A'(t)}{(10-t)^4} + \frac{4A(t)}{(10-t)^5} = \frac{d}{dt}\left[\frac{A(t)}{(10-t)^4}\right]$$

and now integrate both sides between 0 **and** t **(note** $A(0) = 2$**):**

$$\frac{1.5}{(10-t)^3} - \frac{1.5}{10^3} = \frac{A(t)}{(10-t)^4} - \frac{A(0)}{10^4},$$

i.e.,

$$\frac{A(t)}{(10-t)^4} = \frac{2}{10^4} + \frac{1.5}{(10-t)^3} - \frac{1.5}{10^3} = -\frac{13}{10^4} + \frac{1.5}{(10-t)^3}$$

so that

$$A(t) = -13\left(1 - \frac{t}{10}\right)^4 + 1.5(10-t).$$

It is also easy to check that this A **indeed solves the problem.**

7. Solve the IVP $y'' - 2.5y' + y = 0$, $y(0) = 0$, $y'(0) = 1$.

The characteristic polynomial is

$$r^2 - 2.5r + 1 = (r-2)(r-0.5)$$

and its zeros are 2 **and** 0.5 **so that two linearly independent solutions of the equation are given by**

$$y_1(t) = e^{2t} \quad \textbf{and} \quad y_2(t) = e^{0.5t}.$$

Then the solution of the IVP is

$$y(t) = \alpha y_1(t) + \beta y_2(t),$$

where

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}1 & 1\\ 2 & 0.5\end{pmatrix}^{-1}\begin{pmatrix}0\\ 1\end{pmatrix} = -\frac{2}{3}\begin{pmatrix}0.5 & -1\\ -2 & 1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}\frac{2}{3}\\ -\frac{2}{3}\end{pmatrix}.$$

Hence the solution of the IVP is

$$y(t) = \frac{2}{3}\left(e^{2t} - e^{0.5t}\right).$$