30. Show: If $\sum_{k=0}^{\infty} a_k$ is absolutely convergent, then $\sum_{k=0}^{\infty} a_k \sin(kx)$ and $\sum_{k=0}^{\infty} a_k \cos(kx)$ converge uniformly on $\mathbb{R}$.
31. Suppose that $\sum_{k=0}^{\infty} f_k$ converges uniformly, that $\{g_k(x)\}$ is a monotone sequence for each x, and that the sequence $\{\|g_k\|_\infty\}$ is bounded. Show that $\sum_{k=0}^{\infty} f_k g_k$ is uniformly convergent.
32. Prove that $f(x) = \sum_{k=0}^{\infty} a_k x^k$ converges uniformly on $[0,1]$ provided $\sum_{k=0}^{\infty} a_k$ converges. Show that in this case $\lim_{x\to 1-} f(x) = \sum_{k=0}^{\infty} a_k$.
33. Suppose that $\{\| \sum_{k=1}^{n} f_k\|_\infty\}$ is bounded, that $\{g_k(x)\}$ is monotone for each x, and that $g_k \to 0$ uniformly. Show that $\sum_{k=0}^{\infty} f_k g_k$ is uniformly convergent.
34. Prove that $g_1 - g_2 + g_3 - \ldots$ converges uniformly provided $g_1 \geq g_2 \geq \ldots$ and $g_k \to 0$ uniformly.
35. Let $F_n = \sum_{k=1}^{n} f_k$. Prove that $\sum_{k=1}^{\infty} f_k g_k$ converges uniformly provided $\{F_n g_{n+1}\}$ and $\sum_{k=1}^{\infty} F_k(g_k - g_{k+1})$ converge uniformly.
36. Decide whether the following spaces are Banach spaces.
    (a) $\mathbb{R}$ with norm $\|x\| = |x|$;
    (b) $C[0,1]$ with norm $\|x\| = \int_0^1 |x(t)|\mathrm{d}t$;
    (c) the space of sequences that are 0 eventually with the maximal element of a sequence as its norm.
37. Suppose that $\{f_n\}$ is a monotonic function sequence that converges pointwise on $[a,b]$ to f. If f and every f_n are continuous on $[a,b]$, show that $\{f_n\}$ converges uniformly on $[a,b]$ to f.
38. Find examples that show that in each of Theorems 2.8, 2.12, and 2.14 the uniform convergence is not a necessary condition for the claims of the theorems to hold.