56. Find the Fourier coefficients of f on $[-l,l]$ if f is
    (a) even;
    (b) odd.

57. Find the Fourier coefficients of f on $[-\pi,\pi]$ for
    (a) $f(x)=x$;
    (b) $f(x)=|x|$;
    (c) $f(x)=|\sin x|$;
    (d) $f(x)=x^2$;
    (e) $f(x)=\cosh(\alpha x)$, $\alpha\neq 0$.

58. For the set of real-valued polynomials on $[-1,1]$, show that p defined by $p(x)=x$ is orthogonal to every constant function. Next, find a quadratic polynomial that is orthogonal to both p and the constant functions. Finally, find a cubic polynomial that is orthogonal to all quadratic polynomials. Hence construct an orthonormal set with three vectors.

59. Find $\sum_{k=1}^{n}\sin(k\theta)$.

60. For $|a|<1$, find
    (a) $\sum_{n=0}^{\infty}a^n\cos(n\theta)$;
    (b) $\sum_{n=1}^{\infty}a^n\sin(n\theta)$.

61. Let $e_n(x)=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{inx}$, where $x\in(-\pi,\pi)$. Let f be continuous and 2π-periodic on $\mathbb{R}$. Put $f_m=\sum_{n=-m}^{m}\langle f,e_n\rangle e_n$ and $F_m=\frac{1}{m+1}\sum_{k=0}^{m}f_k$.
    (a) Establish the formula $F_m(y)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)K_m(y-x)\mathrm{d}x$, where $K_m(\theta)=\frac{1}{m+1}\sum_{k=0}^{m}\sum_{n=-k}^{k}\mathrm{e}^{in\theta}$ is the so-called Fejér kernel.
    (b) Show that $K_m(\theta)=\frac{1}{m+1}\frac{\sin^2\frac{(m+1)\theta}{2}}{\sin^2\frac{\theta}{2}}$ if $\theta\neq 2\pi n$ for some $n\in\mathbb{Z}$.
    (c) Prove: $F_m(y)-f(y)=\frac{1}{2\pi}\int_{y-\pi}^{y+\pi}[f(x)-f(y)]K_m(y-x)\mathrm{d}x$.
    (d) Draw the graph of K_m for $m\in\{2,5,8\}$.