Problems #3, Math 315, Dr. M. Bohner.Jan 26, 2005. Due Feb 4, 2 pm.

21. Let $f_n(x) = x - \frac{x^n}{n}$. Find the limit function f of $\{f_n\}$ on $[0,1]$ and decide whether $f_n' \to f'$. Does $\int_0^1 f_n(x)\mathrm{d}x \to \int_0^1 f(x)\mathrm{d}x$ hold?
22. Let $f_n(x) = nx(1-x)^n$. Graph f_1, f_2, f_5, and f_{11} on $[0,1]$. Determine the limit function f of $\{f_n\}$ as $n \to \infty$. Is the limit function continuous? Is $f_n' \to f'$ true? How about the integral? Discuss whether $\{f_n\}$ is uniformly convergent on $[0,1]$.
23. Let $f_n(x) = nxe^{-nx}$. Is $\{f_n\}_{n\in\mathbb{N}}$ uniformly convergent on $(0,\infty)$?
24. Assume $\alpha_n \to 0$. Show: $f_n \to f$ uniformly if $|f_n(x) - f(x)| \le \alpha_n$ for all $n \ge N$ and all $x \in E$.
25. Let $f_n(x) = nx/(1+n^2x^2)$ for $x \in [0,1]$. Show that the sequence converges uniformly on $[q,1]$ for any $q \in (0,1)$ but not on $[0,1]$.
26. Prove: If $f_n \to f$ uniformly on E and $g \in \mathrm{B}(E)$, then $f_n g \to fg$ uniformly on E.
27. Prove: If $f_n \to f$ uniformly on E and $|f_n(x)| \ge \alpha > 0$ for all $n \in \mathbb{N}$ and all $x \in E$, then $1/f_n \to 1/f$ uniformly on E.
28. Show: If $\sum_{k=0}^\infty a_k$ is absolutely convergent, then $\sum_{k=0}^\infty a_k \sin(kx)$ and $\sum_{k=0}^\infty a_k \cos(kx)$ converge uniformly on $\mathbb{R}$.
29. Let $F_n = \sum_{k=1}^n f_k$. Prove that $\sum_{k=1}^\infty f_k g_k$ converges uniformly provided $\{F_n g_{n+1}\}$ and $\sum_{k=1}^\infty F_k(g_k - g_{k+1})$ converge uniformly.
30. Suppose that $\sum_{k=0}^\infty f_k$ converges uniformly, that $\{g_k(x)\}$ is a monotone sequence for each x, and that the sequence $\{\|g_k\|_\infty\}$ is bounded. Show that $\sum_{k=0}^\infty f_k g_k$ is uniformly convergent.
31. Suppose that $\{\|\sum_{k=1}^n f_k\|_\infty\}$ is bounded, that $\{g_k(x)\}$ is monotone for each x, and that $g_k \to 0$ uniformly. Show that $\sum_{k=0}^\infty f_k g_k$ is uniformly convergent.
32. Suppose that $g_1 \ge g_2 \ge \dots$ and $g_k \to 0$ uniformly. Prove that $g_1 - g_2 + g_3 - \dots$ converges uniformly.