45. Read Chapter 3 of the book. Work on at least two problems from different sections.
46. Find the solution of the wave equation with Dirichlet conditions (see Theorem 4.1) and
    (a) $\phi(x) = 3\sin\frac{\pi x}{l}$, $\psi(x) = 0$;
    (b) $\phi(x) = 3\sin\frac{\pi x}{l} - 2\sin\frac{3\pi x}{l}$, $\psi(x) = 4\sin\frac{\pi x}{l} + 2\sin\frac{4\pi x}{l}$.
47. Find the solution of the diffusion equation with Dirichlet conditions (see Theorem 4.2) and
    (a) $\phi(x) = 3\sin\frac{\pi x}{l}$;
    (b) $\phi(x) = 3\sin\frac{\pi x}{l} - 2\sin\frac{3\pi x}{l}$.
48. Consider a metal rod ($0 < x < l$), insulated along its sides but not at its ends, which is initially at temperature one everywhere. Suddenly both ends are plunged into a bath of temperature zero. Write the differential equation, boundary conditions, and initial conditions. Write the formula for the temperature $u(x,t)$ at later times. In this problem, you can use the infinite series expansion $\sum\limits_{n=1}^{\infty} \frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{l} = \frac{\pi}{4}$.
49. Find all eigenvalues and eigenfunctions of $f'' + \lambda f = 0$, $f(0) = f(\pi) = 0$. How many zeros inside the interval $(0,\pi)$ does the nth eigenfunction of the problem have?
50. Find all eigenvalues and eigenfunctions of $f'' + \lambda f = 0$, $f(-\pi) = f(\pi)$, $f'(-\pi) = f'(\pi)$. Also show that the eigenfunctions are orthogonal in the sense that $\int\limits_{-\pi}^{\pi} e_1(x)e_2(x)\mathrm{d}x = 0$ whenever e_1 and e_2 are eigenfunctions corresponding to two different eigenvalues.
51. Separate the variables for the equation $tu_t = u_{xx} + 2u$ with $u(0,t) = u(\pi,t) = 0$. Show that the solution of this problem satisfying in addition $u(x,0) = 0$ is not unique.
52. Use the method of separation of variables and discuss the resulting eigenvalue problems for each of the following:
    (a) $u_{xx} + u_{tt} = 0$ ($0 < x < l$, $t > 0$), $u(0,t) = u(l,t) = 0$;
    (b) $u_{xx} + u_{tt} = 0$ ($0 < x < l$, $t > 0$), $u_x(0,t) = u_x(l,t) = 0$;
    (c) $u_{tt} = c^2u_{xx}$ ($0 < x < l$), $u(0,t) = 0$, $u_{tt}(l,t) + ku_x(l,t) = 0$;
    (d) $u_{tt} + a^2u_{xxxx} = 0$ ($0 < x < l$, $t > 0$), $u(0,t) = u(l,t) = u_{xx}(0,t) = u_{xx}(l,t) = 0$.
53. Show that $\cos(nx)$ and $\sin(mx)$ are orthogonal.