18. Show that none of the three transformations introduced in Example 1.7 (i.e., rotation of axes, change of dependent variable, and change of scale) changes the type of the PDE when applied to a second order equation with constant coefficients.

19. Solve the wave equation $u_{tt} = c^2 u_{xx}$, together with the initial conditions

    (a) $u(x,0) = \mathrm{e}^x$ and $u_t(x,0) = \sin x$;

    (b) $u(x,0) = \log(1+x^2)$ and $u_t(x,0) = 4+x$.

20. If both ϕ and ψ are odd functions of x, show that the solution of the initial value problem given in Theorem 2.2 is also odd in x for all times t.

21. Use a method similar to the methods from Theorem 2.1 and Theorem 2.2 (i.e., "factor" the operator) to find the solutions to the following initial value problems:

    (a) $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, $u(x,0) = x^2$, $u_t(x,0) = \mathrm{e}^x$;

    (b) $u_{xx} + 2u_{xt} - 3u_{tt} = 0$, $u(x,0) = \sin x$, $u_t(x,0) = x$;

    (c) $u_{xx} + u_{tt} = 0$, $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$.

22. Find the general solution of the so-called spherical wave equation $u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right)$ by changing variables $v = ur$. Also, find the solution of the spherical wave equation that satisfies $u(r,0) = \phi(r)$ and $u_t(r,0) = \psi(r)$, where ϕ and ψ are differentiable and even.

23. Find the general solution of the nonhomogeneous wave equation $u_{tt} - c^2 u_{xx} = h(x,t)$. Then, determine the solution of this equation that satisfies the initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$.

24. Let $h : \mathbb{R} \to \mathbb{R}$ be a strictly decreasing function. Determine the solution of the so-called Goursat problem, namely of $u_{tt} = c^2 u_{xx}$, $u(x, \frac{x}{c}) = \phi(x)$, $u(x, h(x)) = \psi(x)$.

25. For which functions ϕ and ψ has the problem for the semi-infinite string with a fixed end $u_{tt} = c^2 u_{xx}$ on the right upper quadrant, $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$ for $x \geq 0$, $u(0,t) = 0$ for $t \geq 0$ a unique solution? Find this solution.

26. For which functions ϕ and ψ has the problem for the semi-infinite string with a free end $u_{tt} = c^2 u_{xx}$ on the right upper quadrant, $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$ for $x \geq 0$, $u_x(0,t) = 0$ for $t \geq 0$ a unique solution? Find this solution.