70. For which conditions on $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ are $f(b) = \alpha f(a) + \beta f'(a)$, $f'(b) = \gamma f(a) + \delta f'(a)$ symmetric boundary conditions?

71. Consider an eigenvalue problem $f'' + \lambda f = 0$ with symmetric boundary conditions.
    (a) Show that if $f(b)f'(b) - f(a)f'(a) \le 0$ for all $f : [a,b] \to \mathbb{R}$ satisfying the boundary conditions, then there is no negative eigenvalue.
    (b) Show that the condition in (a) is satisfied for Dirichlet, Neumann, and periodic boundary conditions. In which cases is it satisfied for Robin conditions?

72. Consider the series $\sum_{n=0}^{\infty} (-1)^n x^{2n}$.
    (a) For which $x \in \mathbb{R}$ does the series converge pointwise?
    (b) Does the series converge uniformly on $[-1,1]$?
    (c) Does the series converge in the L^2 sense on $[-1,1]$?

73. Let $\phi(x) = -1 - x$ if $x \in [-1, 0)$ and $\phi(x) = 1 - x$ if $x \in (0,1]$ and $\phi(0) = 0$.
    (a) Find the full Fourier series of ϕ in the interval $(-1,1)$.
    (b) Graph the first five partial sums of the Fourier series (use a computer if you like).
    (c) Does the Fourier series converge in the mean square sense?
    (d) Does the Fourier series converge pointwise?
    (e) Does the Fourier series converge uniformly?

74. Let f be 2π-periodic on $\mathbb{R}$ and assume $\int_{-\pi}^{\pi} |f(x)|^2 \mathrm{d}x < \infty$. Show that both $\int_{-\pi}^{\pi} f(x)\cos(nx)\mathrm{d}x$ and $\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x$ converge to zero as n tends to infinity.

75. Find $\sum_{k=1}^{n} \sin(k\theta)$.

76. For $|a| < 1$, find
    (a) $\sum_{n=0}^{\infty} a^n \cos(n\theta)$;
    (b) $\sum_{n=1}^{\infty} a^n \sin(n\theta)$.

77. Let $e_n(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{inx}$, where $x \in (-\pi, \pi)$. Let f be continuous and 2π-periodic on $\mathbb{R}$. Define $f_m = \sum_{n=-m}^{m} (f, e_n) e_n$ and $F_m = \frac{1}{m+1} \sum_{k=0}^{m} f_k$.
    (a) Establish the formula $F_m(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) K_m(y-x)\mathrm{d}x$, where $K_m(\theta) = \frac{1}{m+1} \sum_{k=0}^{m} \sum_{n=-k}^{k} \mathrm{e}^{in\theta}$ is the so-called Fejér kernel.
    (b) Show that $K_m(\theta) = \frac{1}{m+1} \frac{\sin^2 \frac{(m+1)\theta}{2}}{\sin^2 \frac{\theta}{2}}$ if $\theta \ne 2\pi n$ for some $n \in \mathbb{Z}$.
    (c) Establish the formula $F_m(y) - f(y) = \frac{1}{2\pi} \int_{y-\pi}^{y+\pi} [f(x) - f(y)] K_m(y-x)\mathrm{d}x$.
    (d) Draw the graph of K_m (use a computer if you like) for $m \in \{2, 5, 8\}$.

78. Consider the problem $u_t = k u_{xx}$, $0 < x < l$, $u(x,0) = \phi(x)$ with $u_x(0,t) = u_x(l,t) = \frac{u(l,t) - u(0,t)}{l}$.
    (a) Assume that there are no negative eigenvalues and solve the problem.
    (b) Assume that limits can be taken term by term and find A, B with $\lim_{t\to\infty} u(x,t) = A + Bx$.

79. Read the first three sections of Chapter 8. Work on all of the exercises.