62. Use Problem 60 to find the following infinite series:
(a) \(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\);
(b) \(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \ldots\);
(c) \(\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots\);
(d) \(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \ldots\);
(e) \(\frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{a^2 + n^2}\).

63. Use the previous problem to determine the value of \(\sum_{n=1}^{\infty} \frac{1}{n^2}\).
64. Find the Fourier sine series in \((0, \pi)\) of \(f(x) = \cos x\).
65. Find the complex form of the Fourier series of \(f(x) = e^x\).
66. Let \(V\) be a complex vector space. An inner product on \(V\) is a mapping \((\cdot, \cdot) : V \times V \to \mathbb{C}\) such that for all \(x, y, z \in V\) and all \(\lambda \in \mathbb{C}\): \((x, y) = (y, x)\), \((\lambda x, y) = \lambda (x, y)\), \((x + y, z) = (x, z) + (y, z)\), and \((x, x) > 0\) if \(x \neq 0\).
   (a) Prove \((x, y + z) = (x, y) + (x, z)\) and \((x, \lambda y) = \overline{\lambda}(x, y)\).
   (b) Prove \(\|x\| \geq 0\) and \(\|x\| = 0\) iff \(x = 0\) and \(\|\lambda x\| = |\lambda| \|x\|\).
   (c) Show that \((x, y) = x^T \overline{y}\) is an inner product on \(\mathbb{C}^n\).
   (d) Show that \((f, g) = \int_a^b \overline{f(x)}g(x)dx\) is an inner product on the vector space of all continuous complex-valued functions on \([a, b]\).
67. Let \(V\) be a real vector space with inner product \((\cdot, \cdot)\). We call \(x, y \in V\) orthogonal (write \(x \perp y\)) if \((x, y) = 0\). Also, we put \(\|x\| = \sqrt{(x, x)}\). Prove the following statements. Also draw a picture for the case \(V = \mathbb{R}^2\).
   (a) \((x - y) \perp (x + y)\) iff \(\|x\| = \|y\|\).
   (b) \((x - z) \perp (y - z)\) iff \(\|x - z\|^2 + \|y - z\|^2 = \|x - y\|^2\).
   (c) \(\|x + y\|^2 + \|x - y\|^2 = 2 (\|x\|^2 + \|y\|^2)\).
   (d) \(|(x, y)| \leq \|x\| \|y\|\).
   (e) \(\|x + y\| \leq \|x\| + \|y\|\).
68. Let \(V\) be a real vector space with inner product \((\cdot, \cdot)\). Let \(\{e_i : i \in \mathbb{N}\} \subset V\) be orthonormal, i.e., \((e_i, e_j)\) is zero if \(i \neq j\) and is one if \(i = j\). Let \(n \in \mathbb{N}\), \(x \in V\), and \(\lambda_i \in \mathbb{R}\) for \(i \in \mathbb{N}\). Prove:
   (a) \(\left\| \sum_{i=1}^{n} \lambda_i e_i \right\|^2 = \sum_{i=1}^{n} |\lambda_i|^2\);
   (b) \(\left\| x - \sum_{i=1}^{n} \lambda_i e_i \right\|^2 = \|x\|^2 + \sum_{i=1}^{n} |\lambda_i - c_i|^2 - \sum_{i=1}^{n} |c_i|^2\) with \(c_i = (x, e_i)\);
   (c) \(\lim_{n \to \infty} \sum_{i=1}^{n} \|(x, e_i)\|^2\) exists and is less than or equal to \(\|x\|^2\).
69. For this problem, use the inner products defined earlier for the various cases, respectively.
   (a) Find a set of three orthonormal vectors in \(\mathbb{R}^3\).
   (b) Find \(\alpha\) such that \(c_n(t) = e^{\alpha it}\), \(n \in \mathbb{Z}\) are orthonormal on \([-\pi, \pi]\).
   (c) For the set of real-valued polynomials on \([-1, 1]\), show that \(p(x) = x\) is orthogonal to every constant function. Next, find a quadratic polynomial that is orthogonal to both \(p\) and the constant functions. Finally, find a cubic polynomial that is orthogonal to all quadratic polynomials. Hence construct an orthonormal set with three vectors.