Section 1.1

Introduction and Background

What Is a Differential Equation?

An equation containing one or more unknown functions and their derivatives with respect to one or more independent variables is said to be a *differential equation* (DE).

If a DE contains only ordinary derivatives of one or more unknown functions with respect to a single independent variable, then it is called an *ordinary differential equation* (ODE).

A DE which contains one or more partial derivatives is called a *partial differential equation* (PDE).

Examples of Ordinary Differential Equations

$$y'' - y' + 6y = 0$$

$$\frac{dx}{dt} + \frac{dy}{dt} = 3x - 7$$

$$\frac{dy}{dx} = 1 - \frac{y}{x}$$

$$\frac{d^2y}{dt^2} = -9.8$$

Example of an Equation Which Isn't an ODE

$$(x^3 + 5)' = 3x^2$$

This (correct) equation is not an ODE because it involves the derivative of a known function.

Why Should We Study Differential Equations?

In some cases, we can use mathematical models to simulate an actual experiment. This can be much cheaper and faster than running the experiment and is sometimes necessary.

What Is the Focus of This Course?

In this course, we will concern ourselves with:

- Setting up ordinary differential equations
  - This requires the ability to translate between words and math!
- Solving ordinary differential equations
  - This requires the ability to identify some very specific properties of the equation in order to select a solution method.
  - We will often use the form of the equation to guess the form of the solution and then prove that our guess is – or is not – correct.
  - We will see many techniques for solving ODEs. There are many more which we will not cover.

What Is the Focus of This Course?

In this course, we will concern ourselves with:

- Interpreting the solutions of the ordinary differential equations that we solve.

Example 1 – Setting Up a Basic ODE

When we pour a hot cup of coffee, we know that it will gradually start to cool off (unless its surroundings are at or above the temperature of the coffee). Suppose that the temperature of a cup of coffee changes at a rate which is proportional to the difference between the temperature of the coffee and the temperature of its surroundings. Set up an equation to describe this relationship.

Classification of Differential Equations

First classification: Ordinary vs. Partial

Second classification: Order

- The order of a differential equation is the order of the highest derivative in the equation.

Third classification: Linearity

Definition: Linear ODEs

An n^{th}-order ODE is linear if it can be written in the form

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = f(t)$$

where $a_n(t)$, $a_{n-1}(t)$, …, $a_0(t)$ and $f(t)$ are given (known) functions of t.

In other words, an ODE is linear if the dependent variable and its derivatives appear in additive combinations of their first powers only.

If $f(t) = 0$, then the linear ODE is called homogeneous.

Definition: Nonlinear ODEs

If an ODE is not linear, we call it nonlinear

Example 2 – Classifying ODEs

For each of the following ODEs, determine their order and their linearity. If the ODE is linear, determine whether or not it is homogeneous. Also identify the independent and dependent variables.

Example 2 – Classifying ODEs

Equation	Order	Linear	Homog.	Ind. Var.	Dep. Var.
$\frac{dy}{dt} = 3t^2$					
$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 6 = 0$					
$y'' + 2y' - 6y^2 = 0$					
$t^2y'' - 4ty' + 4y = e^t$					
$h'(t) = 3h(t) + \cos(h(t))$					
$t^5x''' - 5tx'' + 3x' = 0$					
$yy' + 4ty = t^2$					
$\frac{dx}{dt} + \frac{dy}{dt} = 3x - 7$					