Exam 3 Review Sheet

Math 3304

7.5-7.9 - Solving Initial Value Problems with Laplace Transforms

Problem 1. Find the solution of the initial value problem

$$y'' + 2y' + 2y = \delta(t-5), \quad y(0) = 0, \quad y'(0) = 0.$$

Then compute the value of the solution, accurate to five decimal places, when $t = 6$.

Problem 2. Solve the initial value problem

$$y'' - 4y' + 5y = \delta(t-2\pi), \quad y(0) = 0, \quad y'(0) = 0.$$

Express your final answer without any unit step functions.

Problem 3. Solve the integrodifferential equation

$$y'(t) = 1 - \int_0^t e^{-2\tau} y(t-\tau)d\tau, \quad y(0) = 1.$$

Problem 4. Find the solution of the integro-differential equation

$$y'(t) + 3\int_0^t e^{-4r}\, y(t-r)\, dr = 1, \quad y(0) = 0.$$

Problem 5. Solve the initial value problem

$$y'' + 4y = 1 - u(t-3\pi), \quad y(0) = 0, \quad y'(0) = 0.$$

Problem 6. Solve the initial value problem

$$y'' - 2y' + 2y = \delta(t-2), \quad y(0) = 1, \quad y'(0) = -2.$$

Problem 7. Solve the integral equation

$$y(t) + \int_0^t e^{-\tau}\, y(t-\tau)\, d\tau = 1.$$

Problem 8. (a) Find the solution $y = y(t)$ of the initial value problem

$$y'' + 3y' + 2y = \delta(t-5), \quad y(0) = 0, \quad y'(0) = 1.$$

(b) Which is greater, $y(6)$ or $y(1)$? Justify your answer.

Problem 9. Solve the integral equation

$$y(t) = 3t^2 - e^{-t} - \int_0^t y\,(u)e^{t-u}\,du.$$

Problem 10. Find the solution of the initial value problem

$$y'' + 4y = 2 - 4u(t - \frac{\pi}{2}), \quad y(0) = 0, \quad y'(0) = 0.$$

Problem 11. Find the solution of the initial value problem

$$y'' + y = \frac{1}{\pi}\delta(t - \pi), \quad y(0) = 0, \quad y'(0) = -1.$$

Write your solution as a piecewise defined function and sketch its graph on the interval

$$0 \le t \le 3\pi.$$

Problem 12. Solve the integral equation

$$y(t) + 5\int_0^t y\,(t - r)\cos(2r)\,dr = e^{-3t}.$$

Problem 13. Compute the inverse Laplace transform of

$$F(s) = \frac{2s^2 + 6s + 12}{s(s-2)(s+2)}.$$

Problem 14. (a) Solve the initial value problem

$$y'' + 4y = g(t), \quad y(0) = 1, \quad y'(0) = 0,$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 3\pi, \\ 1, & 3\pi \le t < \infty. \end{cases}$$

(b) Which is greater, $y(2\pi)$ or $y(5\pi)$? Justify your answer.

Problem 15. Solve the initial value problem

$$y^{(4)} - y = 2\delta(t - 1), \quad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0.$$

Problem 16. (a) Find the solution of the initial value problem

$$y'' + y = g(t), \quad y(0) = 1, \quad y'(0) = -1,$$

where

$$g(t) = \begin{cases} 1 - t, & t < 1, \\ 0, & t \geq 1. \end{cases}$$

(b) Draw the graph of the solution on the interval

$$0 \leq t \leq 3\pi.$$

Problem 17. Solve the integro-differential equation

$$y'(t) + 2y(t) - 2\int_0^t y\,(\xi)\sin(t - \xi)\,d\xi = -\sin(t),$$

subject to the initial condition

$$y(0) = 1.$$

Problem 18. (a) Solve the initial value problem

$$y'' + 4y = \delta(t - \pi), \quad y(0) = 1/2\,, \quad y'(0) = 0.$$

(b) Which is greater, $y(\pi)$ or $y(3\pi/2)$? Justify your answer.

Problem 19. Solve the initial value problem

$$y'' + 4y' + 13y = h(t), \quad y(0) = y'(0) = 0,$$

where

$$h(t) = \begin{cases} 0, & 0 \leq t < \pi, \\ 13, & \pi \leq t < \infty. \end{cases}$$

Problem 20. Find the solution $y(t)$ of the integral equation

$$y(t) - 9\int_0^t (t - \tau)y(\tau)\,d\tau = 9t^2.$$

Problem 21. Solve the initial value problem

$$y'' + 3y' + 2y = u(t - 10), \quad y(0) = y'(0) = 0$$

Problem 22. Find the solution of the integral equation

$$y(t) - \int_0^t y\,(\tau)sin(t - \tau)d\tau = \delta(t - \pi)cos(t)$$

Problem 23. Solve the initial value problem

$$y' + 3\int_0^t e^{-4(t-\tau)}\,y(\tau)d\tau = 0, \qquad y(0) = 1$$

Problem 24. Solve the initial value problem

$$y''(t) + 2y'(t) + 5y(t) = 2\delta(t - \frac{\pi}{2})sin(t), \qquad y(0) = 1, y'(0) = -1$$

Problem 25. Solve the initial value problem

$$y'' + 9y = f(t), y'(0) = 1$$

where

$$f(t) = \begin{cases} 3 & 0 \le t < 2 \\ 0 & 2 \le t < \infty \end{cases}$$

Problem 26. Solve the Integro-differential Equation

$$y' + \int_0^t c\,os(t - \tau)y(\tau)d\tau = 1, \quad y(0) = 0$$

Problem 27. Given that $\mathcal{L}(tsin(at)) = \frac{2as}{(s^2+a^2)^2}$, solve the IVP

$$y'' + 9y = \delta(t - 3\pi) + cos(3t), \quad y(0) = y'(0) = 0$$

Calculate the value of $y(\frac{2\pi}{3})$.

Problem 28. Find the solution of the integral equation

$$y(t) - \int_0^t \tau\, y(t - \tau)d\tau = e^t$$

Problem 29. Solve the initial value problem

$$y'' + y = u_\pi(t), \qquad y(0) = 0, \quad y'(0) = 1.$$

Problem 30. Solve the integro-differential equation

$$2y'(t) = \int_0^t (t - \tau)^2 y(\tau)\, d\tau - 2t$$

with

$$y(0) = 1.$$

Problem 31. Solve the initial value problem

$$y'' + 3y' + 2y = g(t), \qquad y(0) = 0, \qquad y'(0) = 0,$$

where

$$g(t) = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t. \end{cases}$$

Problem 32. Solve the integral equation

$$y(t) + 4\int_0^t (t-\tau)y(\tau)\,d\tau = t\,u_1(t) - t.$$

Problem 33. Solve the initial value problem

$$y'' - 4y' + 5y = \delta(t-\pi), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Problem 34. Find the solution of the integral equation

$$y(t) - \int_0^t \sin(t-\tau)y(\tau)\,d\tau = t.$$

Problem 35. Use the Laplace transform to solve the initial value problem

$$y' - y = f(t), \qquad y(0) = 1,$$

where

$$f(t) = \begin{cases} 0, & 0 \le t < 4, \\ e^{3(t-4)}, & t \ge 4. \end{cases}$$

Problem 36. Solve the integral equation

$$y(t) + 3\int_0^t y(\tau)\sin(t-\tau)\,d\tau = t.$$

Problem 37. Solve the initial value problem

$$y'' + 2y' + 2y = \delta(t-\pi), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Problem 38. Find $g(t)$, which is the following inverse Laplace transform:

$$g(t) = \mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{s^2+4}\right\}$$

Problem 39. Make use of Laplace transform to solve the following initial value problem:

$$y'' + 3y' + 2y = 1, \quad y(0) = 0, \quad y'(0) = 0$$

Problem 40. Solve the following integral equation:

$$y(t) - \int_0^t e^{-(t-v)}\,y(v)\,dv = u(t-1)$$

Problem 41. Solve the given initial value problem using Laplace transforms:

$$y'' - y' - 2y = 0, \quad y(0) = -2, \quad y'(0) = 5$$

Problem 42. Find the solution of the integral equation:

$$y(t) - \int_0^t \sin(t-\tau)y(\tau)\,d\tau = t$$

Problem 43. Solve the initial value problem:

$$2y'' + 4y' + 10y = \delta(t-5), \quad y(0) = 0, \quad y'(0) = 0$$

Problem 44. Use Laplace Transform method to solve the initial value problem:

$$y'' + 4y = 2; \quad y(0) = 0, \quad y'(0) = 0$$

Problem 45. Solve the following initial value problem:

$$y'' - 4y' + 4y = \delta(t-4), \quad y(0) = 0, \quad y'(0) = 1$$

Problem 46. Solve the initial value problem:

$$y' - 5y = 1 + u(t-3), \quad y(0) = 0$$

Problem 47. Solve the equation:

$$y(t) - \int_0^t e^{-v}\, y(t-v)\, dv = 3$$

Problem 48. Solve the initial value problem:

$$y'' + y = 4\delta(t-2\pi), \quad y(0) = 1, \quad y'(0) = 0$$

Problem 49. Use the Laplace transform method to solve the initial value problem:

$$y'' - 2y' - 3y = 8e^t, \quad y(0) = 0, \quad y'(0) = 0$$

Problem 50. Solve the initial value problem:

$$y'' + 2y' + y = \delta(t-3), \quad y(0) = 1, \quad y'(0) = -1$$

Problem 51. Find the following information for the function:

$$g(t) = e^{3(t-2)}u(t-2)$$

Determine the values of $g(1)$, $g(3)$, and the Laplace transform $\mathcal{L}\{g(t)\}$.

Problem 52. Solve the initial value problem:

$$y' + 3y = 1 + u(t-2), \quad y(0) = 0$$

Problem 53. Solve the initial value problem:

$$y'' - 2y' + 5y = 2\delta(t-3), \quad y(0) = 0, \quad y'(0) = 2$$

Problem 54. Solve the integro-differential equation

$$y' + 2y - \int_0^t 2\, y(\tau)\sin(t-\tau)\, d\tau = -\cos t$$

subject to the initial condition $y(0) = 0$.

Problem 55. Solve the initial value problem

$$y'' + y = \delta(t - \pi), \quad y(0) = 0, \quad y'(0) = 1$$

and compute the numerical values of $y\left(\frac{\pi}{2}\right)$ and $y\left(\frac{3\pi}{2}\right)$.

Problem 56. Solve the integral-differential equation using the method of Laplace transforms:

$$y'(t) + 2y(t) - 2\int_0^t y\,(v)\sin(t - v)\,dv = \cos(t), \quad y(0) = 0$$

Problem 57. Solve the initial value problem using the method of Laplace transforms:

$$y''(t) - 4y'(t) + 5y(t) = \delta(t - 3), \quad y(0) = 0, \quad y'(0) = 1$$

Problem 58. Solve the initial value problem using the method of Laplace transforms:

$$y'' + 5y' + 6y = t\delta(t - 3), \quad y(0) = 0, \quad y'(0) = 0$$

Problem 59. Solve the integral equation using the method of Laplace transforms:

$$(\dagger) \quad y(t) + \int_0^t (t - v)y(v)\,dv = 2t$$

Problem 60. Solve the initial value problem using the method of Laplace transforms:

$$y'' + 2y' + 2y = g(t), \quad y(0) = y'(0) = 0$$

where $g(t) = 0$ for $t \le 2$ and $g(t) = 1$ for $t > 2$.

9.1/5.1 - Introductions to Systems of Linear Differential Equations, Interconnected Fluid Tanks

Problem 61. Two tanks initially hold 200 gallons of pure water each. A mixture with concentration 5 oz/gal flows into Tank 1 at 5 gal/min. The mixture drains from Tank 1 into Tank 2 at 2 gal/min and into the environment at 1 gal/min. Tank 2 drains into the environment at 2 gal/min. Let $Q_1(t)$ and $Q_2(t)$ denote the salt (oz) in Tanks 1 and 2. Set up, but do not solve, a system of differential equations with initial conditions modeling the process.

Problem 62. Consider two interconnected tanks. Tank 1 initially contains 40 grams of sugar dissolved in 60 liters of water, and Tank 2 initially contains 50 liters of water with 25 grams of sugar. Water containing 2 grams of sugar per liter flows into Tank 2 at 3 liters per minute. The mixture drains from Tank 2 at 4 liters per minute, with 1.5 liters per minute flowing into Tank 1 and the rest leaving the system. The mixture in Tank 1 flows back into

Tank 2 at 1.5 liters per minute. If $S_1(t)$ and $S_2(t)$ denote the amounts of sugar in Tanks 1 and 2, respectively, set up, but do not solve, an initial value problem modeling the process.

Problem 63. Two very large tanks initially hold 100 gallons of pure water each. A mixture with concentration 3 oz/gal flows into Tank 1 at 5 gal/min. The mixture drains from Tank 1 into the environment at 2 gal/min and into Tank 2 at 3 gal/min. Another mixture with concentration 4 oz/gal flows into Tank 2 at 2 gal/min. Tank 2 drains into the environment at 2 gal/min. If $Q_1(t)$ and $Q_2(t)$ denote the amounts of salt in Tanks 1 and 2, respectively, set up, but do not solve, the differential equations and initial conditions modeling the process.

Problem 64. (a) Transform the fourth-order differential equation

$$y^{(4)} - y'' - y = \sin(t^2)$$

into an equivalent system of first-order differential equations.

(b) Express this first-order system in vector-matrix notation.

Do not attempt to solve either the system or the original differential equation.

Problem 65. Consider a system of two interconnected tanks. Tank 1 initially contains 30 gal of water and 25 oz of salt, and Tank 2 initially contains 20 gal of water and 15 oz of salt. Water containing 1 oz/gal of salt flows into Tank 1 at a rate of 1.5 gal/min. The mixture flows from Tank 1 to Tank 2 at a rate of 3 gal/min. Water containing 3 oz/gal of salt also flows into Tank 2 at a rate of 1 gal/min (from the outside). The mixture drains from Tank 2 at a rate of 5 gal/min, of which some flows back into Tank 1 at a rate of 1.5 gal/min, while the remainder leaves the system.

(a) Set up, but do not solve, an initial value problem modeling the amount of salt in Tank 1 and Tank 2, respectively, at any time t.

(b) For which times t is your model in part (a) valid? Explain why this is so.

Problem 66. Consider two tanks holding initially 100 gallons of pure water each. A mixture of salt and water at a concentration of 4 ounces per gallon flows into Tank 1 at a rate of 5 gallons per minute. The well-stirred mixture in Tank 1 drains into the environment at a rate of 3 gallons per minute, and from Tank 1 into Tank 2 at a rate of 2 gallons per minute. Another mixture of salt and water at a concentration of 5 ounces per gallon flows into Tank 2 at a rate of 2 gallons per minute. The well-stirred mixture in Tank 2 drains into the environment at a rate of 2 gallons per minute.

If $Q_1(t)$ and $Q_2(t)$ denote the amounts of salt in ounces at time t in Tanks 1 and 2, respectively, SET UP, BUT DO NOT SOLVE, the differential equations and initial conditions that model the flow process.

Problem 67. Consider two interconnected tanks. Tank 1 initially contains 25 gal of water and 20 oz of salt, and Tank 2 initially contains 15 gal of water and 10 oz of salt. Water containing 2 oz/gal of salt flows into Tank 1 at a rate of 2.5 gal/min. The mixture flows from

Tank 1 to Tank 2 at a rate of 3 gal/min. Water containing 4 oz/gal of salt also flows into Tank 2 at a rate of 2 gal/min from outside. The mixture drains from Tank 2 at a rate of 5 gal/min, of which some flows back into Tank 1 at a rate of 2.5 gal/min, while the remainder leaves the system.
Set up, BUT DO NOT SOLVE, an initial value problem which models the amount of salt in each tank at all future times.

Problem 68. Consider a system of two interconnected tanks. Tank 1 contains initially 20 gallons of pure water, and Tank 2 initially contains 5 gallons of water and 20 oz of salt. A mixture of salt and water at a concentration of 5 ounces per gallon flows into Tank 1 at a rate of 2 gallons per minute. The well-stirred mixture in Tank 1 drains into Tank 2 at a rate of 3 gallons per minute. A mixture of salt and water at a concentration of 1 ounce per gallon flows into Tank 2 at a rate of 4 gallons per minute, and the well-stirred mixture in Tank 2 drains at a rate of 7 gallons per minute, of which some flows back into Tank 1 at a rate of 2 gallons per minute, while the remainder leaves the system.
If $Q_1(t)$ and $Q_2(t)$ denote the amounts of salt in ounces at time t in Tanks 1 and 2, respectively, SET UP, BUT DO NOT SOLVE, an initial value problem modeling the amount of salt in Tank 1 and Tank 2 at any time t .

Problem 69. Two interconnected tanks are each initially filled with 100 gallons of pure water. A mixture containing 2 pounds of salt per gallon of water is pumped into Tank 1 at a rate of 3 gal/min, and into Tank 2 at a rate of 2 gal/min. The well-stirred mixture in Tank 1 flows through a pipe into Tank 2 at a rate of 4 gal/min. Also, the well-stirred mixture in Tank 2 flows through another pipe into Tank 1 at a rate of 5 gal/min and into the environment at a rate of 1 gal/min.

1. Draw and label a simple diagram of the system.
2. Find the volume of each tank at any time $t > 0$ (in minutes).
3. With Q_1 and Q_2 denoting the salt content in tanks 1 and 2, in pounds, set up, but DO NOT SOLVE, the differential equations modeling the system.

Problem 70. Consider an interconnected system of two tanks, Tank 1 and Tank 2.

- Tank 1 initially contains 1 gal of water and 2 oz of salt.
- Tank 2 initially contains 1 gal of water and no salt.

The mixture flows from Tank 1 to Tank 2 at a rate of 1 gal/min. The mixture flows from Tank 2 to Tank 1 at a rate of 1 gal/min. Let $x(t)$ and $y(t)$ be the amount of salt (in oz) in Tank 1 and Tank 2, respectively, at time t.

Set up a system of differential equations, including initial conditions, which models the flow process.

9.2-9.3 - Systems and Linear Algebra Problems

Problem 71. If

$$A(t) = \begin{pmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{pmatrix},$$

compute

$$\frac{d}{dt}(A^{-1}(t)).$$

Problem 72. If

$$A(t) = \begin{pmatrix} 2e^{2t} & e^{-t} \\ 2e^{t} & 3e^{-2t} \end{pmatrix},$$

find

$$\frac{d}{dt}(A^{-1}(t)).$$

Problem 73. Verify that the matrix function

$$X(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$$

satisfies the differential equation

$$X' = AX, \quad A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}.$$

Problem 74. Let

$$A(t) = \begin{pmatrix} \sin t & -\cos t \\ -\cos t & -\sin t \end{pmatrix}.$$

Does

$$\frac{d}{dt}(A^2)$$

equal $2A\frac{dA}{dt}$, or $2\frac{dA}{dt}A$, or $A\frac{dA}{dt} + \frac{dA}{dt}A$, or none of these? Show work supporting your answer.

Problem 75. Are the vectors

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Linearly independent? Show work to justify your answer

Problem 76. a) Transform the following differential equation into a system of first order differential equations:

$$y^{(4)} + 3y^{(3)} - y' + y = t^{-1}sin(t)$$

b) Rewrite the system in matrix/vector form

Problem 77. Given that $x_1(t) = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$ and $x_2(t) = \begin{bmatrix} e^t \\ 3e^{-t} \end{bmatrix}$ are two linearly independent solutions of

$$x' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} x$$

Find the fundamental matrix, X, such that $X(0) = I$

Problem 78. Let $x_1 = y,\ x_2 = y',\ x_3 = y'',\ x_4 = y'''$. Then solve the following differential equation by transforming it into a system of first-order differential equations:

$$y^{(4)} + 2y'' + 6y = t^3.$$

Problem 79 (11). Find constants α and β so that

$$\mathbf{x}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$$

is a solution of

$$\mathbf{x}'(t) = \begin{pmatrix} \alpha & -2 \\ 2 & \beta \end{pmatrix} \mathbf{x}(t).$$

Problem 80 (11). Determine whether the following vectors are linearly dependent or independent. If they are linearly dependent, find a linear relation among them.

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Problem 81.

1. Transform the following differential equation

$$y^{(4)} - e^t y''' + 2y = t\cos(t)$$

   into a system of first order differential equations.

2. Write the system that you obtain in part (a) in matrix-vector form.

Problem 82. Complete the problems below.

1. Express the differential equations

$$x'' + 3x' - y' + 2y = 0, \qquad y'' + x' + 3y' + y = 0$$

as a first order system in normal form. Then write the system in the form

$$\mathbf{x}' = A\mathbf{x}.$$

2. Determine whether the following vector functions are linearly dependent or linearly independent. Justify your answer.

$$\mathbf{x}_1 = \begin{pmatrix} e^t \\ 0 \\ 2e^t \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} -e^t \\ e^t \\ 0 \end{pmatrix}, \qquad \mathbf{x}_3 = \begin{pmatrix} e^t \\ e^t \\ 4e^t \end{pmatrix}.$$

Problem 83. Rewrite the scalar higher order linear differential equation

$$y''' + 2y'' + 4y = 11t$$

in matrix-vector form $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$.

Problem 84. Express the system of differential equations

$$\begin{aligned} x' &= 2x + 3y + z \\ y' &= 4z - 5y \\ z' &= 3y - 2x \end{aligned}$$

in the form

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

and determine the values for the entries of matrix A.

Problem 85. Express the following system of differential equations

$$\begin{cases} \dot{x}_1 = 3x_1 + 2x_2 \\ \dot{x}_2 = 4x_1 + 5x_2 \end{cases}$$

in the matrix form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and determine the values of $a, b, c,$ and d.

Problem 86. Let $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Find a matrix A such that the system of differential equations

$$\begin{cases} x' = y + 2x + z \\ y' = 2z - x \\ z' = 4x \end{cases}$$

is equivalent to $\mathbf{u}' = A\mathbf{u}$.

Problem 87. Express the differential equation

$$y'''(t) + 3y'(t) - 20y(t) = 0$$

as a system of first order differential equations. (Do NOT write the system in matrix form.)

Problem 88. Express the system of differential equations

$$\begin{aligned} x' &= 4x + y + 2z \\ y' &= -3z - 7y \\ z' &= 5y - 6x \end{aligned}$$

in matrix form by finding the matrix A so that:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Problem 89. Express the system of differential equations

$$\begin{aligned} x_1' &= x_1 + 2x_3 \\ x_2' &= 9x_3 - 6x_2 + 3x_1 \\ x_3' &= -4x_2 + 5x_3 \end{aligned}$$

in the form

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

and determine the values for the entries of matrix A.

Problem 90. Express the system of differential equations

$$\begin{aligned} x' &= x + y + z \\ y' &= 2z - x \\ z' &= 4y \end{aligned}$$

in matrix form by finding the matrix A so that:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Problem 91. Express the damped mass-spring oscillator equation

$$(*) \quad my'' + by' + ky = 0$$

as a system of first order differential equations. Do NOT write the system in matrix form.

9.4 - 9.5 - Homogeneous Systems with Constant Coefficients

Problem 92. Find the general solution of the system

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}.$$

Problem 93. Find the solution of the initial value problem

$$\begin{aligned} x_1' &= x_1 - x_2, \quad x_1(0) = 1, \\ x_2' &= x_1 + x_2, \quad x_2(0) = 2. \end{aligned}$$

Problem 94. Find the solution of the system

$$\begin{aligned} \frac{dx}{dt} &= x + 6y, \\ \frac{dy}{dt} &= x - 4y, \end{aligned} \quad x(0) = 5, \quad y(0) = 9.$$

Problem 95. Solve the initial value problem

$$\begin{aligned} x' &= x + y, \\ y' &= 4x + y, \end{aligned} \quad x(0) = 2, \quad y(0) = 0.$$

Problem 96. Find the general solution of the system

$$\begin{aligned} x' &= 3x + 6y, \\ y' &= -x - 2y. \end{aligned}$$

Problem 97. Find the general solution of

$$x' = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix} x$$

Problem 98. Solve the initial value problem

$$\begin{aligned} x' &= -5x + y, \quad x(0) = 2, \\ y' &= -3x - y, \quad y(0) = -1. \end{aligned}$$

Describe the behavior of the solution as $t \to \infty$.

Problem 99. (a) Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(b) Describe the behavior of the solution as $t \to \infty$.

Problem 100. Find two linearly independent vector solutions of

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix},$$

and calculate their Wronskian.

Problem 101. Given that $\Phi(t) = \begin{bmatrix} e^{-t} & e^{t} \\ 3e^{-t} & e^{t} \end{bmatrix}$ is a fundamental matrix for

$$x' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$$

Solve the initial value problem

$$x' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Problem 102. Solve the initial value problem

$$\begin{aligned} x'(t) &= -2x - y \\ y'(t) &= -7x + 4y \end{aligned}$$

with $x(0) = 1, y(0) = 3$

Problem 103. Find the general solution of the system of differential equations

$$x' = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} x$$

Problem 104. Find the general solution of the system of differential equations

$$x' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} x$$

Problem 105. Find the general solution of the system

$$\frac{dx_1}{dt} = -2x_2 + x_1, \qquad \frac{dx_2}{dt} = 3x_1 - 4x_2.$$

Problem 106. Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Problem 107. Find the general solution of the following system of differential equations:

$$\begin{cases} x_1'(t) = x_1(t) + 2x_2(t), \\ x_2'(t) = 2x_1(t) + x_2(t). \end{cases}$$

Problem 108. Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & 2 \\ 10 & 1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Problem 109. Find the general solution to the differential equation system:

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} -1 & 1 \\ 8 & 1 \end{pmatrix}$$

where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Problem 110. Find the general solution of the linear system $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

Problem 111. Find the general solution of the differential equation system:

$$\vec{x}' = A\vec{x}, \quad A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

Problem 112. Find the general solution of the differential equation system:

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

Problem 113. Find the general solution of the differential equation system:

$$\vec{x}' = A\vec{x}, \quad A = \begin{bmatrix} -1 & 1 \\ 8 & 1 \end{bmatrix}$$

Problem 114. Find the general solution of the differential equation system:

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Problem 115. Given that $\mathbf{x} = \begin{bmatrix} e^{6t} \\ e^{6t} \end{bmatrix}$ is a solution of the following system

$$\mathbf{x}' = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix} \mathbf{x},$$

find the general solution of system (1).

Problem 116. Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix}$$

Problem 117. Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$.