Final Exam Sheet

Math 3304

Note: The final exam is comprehensive. It will cover all of the topics from exam 1, 2 and 3 plus two additional topics (section 9.6 and 9.7).

9.6 - Complex Eigenvalues / Eigenvectors

Problem 1. Solve the initial value problem $\mathbf{x}' = \begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix}\mathbf{x}, \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Problem 2. Solve the initial value problem

$$\begin{aligned} x' &= -3x + 2y, \quad x(0) = 0 \\ y' &= -x - y, \quad y(0) = 1 \end{aligned}$$

Problem 3. Solve the system $u' = 2u + 8v$, $v' = -u - 2v$, subject to the initial conditions $u(0) = 2, v(0) = -1$.

Problem 4. Find the solution of the system

$$x' = \begin{bmatrix} 3 & 9 \\ -2 & -3 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

Problem 5. Find a real fundamental matrix of the following system of differential equations

$$\begin{aligned} x_1'(t) &= 2x_1(t) + 8x_2(t) \\ x_2'(t) &= -x_1(t) - 2x_2(t) \end{aligned}$$

Problem 6. Solve the initial value problem

$$x' = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Problem 7. Find the general solution to the differential equation system

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Here $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Problem 8. Find the general solution to the differential equation system

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Problem 9. Find the general solution of the differential equation system

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Problem 10. Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

Problem 11. Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

9.7 - Nonhomogeneous Systems

Problem 12. Given that $\mathbf{x}(t) = c_1 \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} 2e^{-2t} \\ 3e^{-2t} \end{pmatrix}$ is the general solution for the homogeneous system $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}$, solve the nonhomogeneous initial value problem $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^{-t}, \mathbf{x}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

Problem 13. Use the method of variation of parameters for systems to solve the system

$$\begin{cases} \dfrac{dx}{dt} = -y + \sec(t) \\ \dfrac{dy}{dt} = x \end{cases}$$

given that the general solution to the associated homogeneous system is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} + c_2 \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}.$$

Problem 14. Given that $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}$ and $\mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$ form a fundamental set of solutions for $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$, solve the initial value problem $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} te^{2t} \\ te^{2t} \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$.

Problem 15. Find the general solution of the system $x' = 2x + 3y - 7$, $y' = -x - 2y + 5$.

Problem 16. Let $A = \begin{pmatrix} -1/10 & 3/40 \\ 1/10 & -1/5 \end{pmatrix}$. Given that $\Psi(t) = \begin{pmatrix} 3e^{-t/20} & -e^{-t/4} \\ 2e^{-t/20} & 2e^{-t/4} \end{pmatrix}$ is a fundamental matrix for the homogeneous system $\mathbf{x}' = A\mathbf{x}$, find the general solution of the nonhomogeneous system $\mathbf{x}' = A\mathbf{x} + \begin{pmatrix} 6 \\ 12 \end{pmatrix}$.

Problem 17. Find the general solution of $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Problem 18. Let $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ and $\mathrm{f}(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$. Given that the fundamental matrix for the system $\mathrm{x}' = A\mathrm{x}$ is $X(t) = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}$, find the general solution of the nonhomogeneous system $\mathrm{x}' = A\mathrm{x} + \mathrm{f}(t)$

Problem 19. Given that $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$ is a solution of the homogeneous system $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$, find the general solution of the nonhomogeneous differential equation system

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Problem 20 (25). Find a particular solution of the nonhomogeneous system

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2e^{-t} \\ -e^{-t} \end{pmatrix}.$$

You may use that

$$\mathbf{x} = C_1\begin{pmatrix} e^{t} \\ -3e^{t} \end{pmatrix} + C_2\begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$$

is the general solution of the associated homogeneous system.

Problem 21. Find the general solution of the nonhomogeneous system

$$x' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} e^{t} \\ e^{-t} \end{bmatrix}.$$

The fundamental matrix for the associated homogeneous system is given by:

$$\begin{bmatrix} e^{-t} & e^{3t} \\ -e^{-t} & 3e^{3t} \end{bmatrix}$$

Problem 22. Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}.$$

You may use that

$$X(t) = \begin{pmatrix} e^{2t} & e^{3t} \\ -e^{2t} & -2e^{3t} \end{pmatrix}$$

is a fundamental matrix of the associated homogeneous system.

Problem 23. Use Variation of Parameters to find a particular solution to the system

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} = A\mathbf{x} + \mathbf{f}$$

given that the corresponding homogeneous system produces a fundamental matrix

$$X(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix}$$

Problem 24. Find a particular solution of the nonhomogeneous system

$$\mathbf{x}' = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t},$$

where

$$\Phi(t) = \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix}$$

is a fundamental matrix of the associated homogeneous system.

Problem 25. Given that $\Psi(t) = \begin{pmatrix} e^{2t} & e^{3t} \\ -e^{2t} & -2e^{3t} \end{pmatrix}$ is the fundamental matrix of the homogeneous system $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}\mathbf{x}$, find the general solution of:

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

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Note: The final exam is comprehensive. It will cover all of the topics from exam 1, 2 and 3 plus two additional topics (section 9.6 and 9.7).

9.6 - Complex Eigenvalues / Eigenvectors

Problem 1. Solve the initial value problem $\mathbf{x}' = \begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix}\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Solution 1. The system is $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix}$. The eigenvalues are found via $\det(A - \lambda I) = 0$:

$$\det\begin{pmatrix} -\lambda & -1/2 \\ 2 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0 \Longrightarrow \lambda = \pm i.$$

For $\lambda = 0 + 1i$, we have $\alpha = 0$ and $\beta = 1$. We find the eigenvector $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ by solving $(A - iI)\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} -i & -1/2 \\ 2 & -i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow -iv_1 - \frac{1}{2}v_2 = 0.$$

Letting $v_1 = 1$, we get $v_2 = -2i$. Thus, $\mathbf{v} = \begin{pmatrix} 1 \\ -2i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ -2 \end{pmatrix}$, so $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$. Using the formula for real linearly independent solutions in the complex eigenvalue case:

$$\begin{aligned}
\mathbf{x}_1 &= e^{0t}\cos(t)\mathbf{a} - e^{0t}\sin(t)\mathbf{b} = \cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin(t)\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} \cos t \\ 2\sin t \end{pmatrix} \\
\mathbf{x}_2 &= e^{0t}\sin(t)\mathbf{a} + e^{0t}\cos(t)\mathbf{b} = \sin(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos(t)\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} \sin t \\ -2\cos t \end{pmatrix}.
\end{aligned}$$

The general solution is $\mathbf{x}(t) = c_1\begin{pmatrix} \cos t \\ 2\sin t \end{pmatrix} + c_2\begin{pmatrix} \sin t \\ -2\cos t \end{pmatrix}$. Applying $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -2 \end{pmatrix} \Longrightarrow c_1 = 1,\ c_2 = -1.$$

The solution to the initial value problem is:

$$\mathbf{x}(t) = \begin{pmatrix} \cos t - \sin t \\ 2\sin t + 2\cos t \end{pmatrix}.$$

Problem 2. Solve the initial value problem

$$\begin{aligned} x' &= -3x + 2y, \quad x(0) = 0 \\ y' &= -x - y, \quad y(0) = 1 \end{aligned}$$

Solution 2. Write the system in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Find the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix} \\ &= (-3-\lambda)(-1-\lambda) + 2 \\ &= \lambda^2 + 4\lambda + 5. \end{aligned}$$

Thus,

$$\lambda = -2 \pm i.$$

For $\lambda_1 = -2 + i$,

$$\begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This leads to the equation

$$(-1-i)v_1 + 2v_2 = 0.$$

Taking $v_1 = 2$, we get

$$v_2 = 1 + i.$$

Therefore, an eigenvector is

$$V_1 = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}.$$

Now write

$$e^{(-2+i)t} \begin{bmatrix} 2 \\ 1+i \end{bmatrix} = e^{-2t}(\cos(t) + i\sin(t)) \begin{bmatrix} 2 \\ 1+i \end{bmatrix}.$$

Multiplying out,

$$e^{(-2+i)t} \begin{bmatrix} 2 \\ 1+i \end{bmatrix} = e^{-2t} \begin{bmatrix} 2\cos(t) + 2i\sin(t) \\ \cos(t) - \sin(t) + i(\sin(t) + \cos(t)) \end{bmatrix}.$$

Therefore, the two real-valued solutions are the real and imaginary parts:

$$\mathbf{x}_1(t) = e^{-2t} \begin{bmatrix} 2\cos(t) \\ \cos(t) - \sin(t) \end{bmatrix},$$

and

$$\mathbf{x}_2(t) = e^{-2t}\begin{bmatrix} 2\sin(t) \\ \sin(t)+\cos(t) \end{bmatrix}.$$

Therefore, the general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-2t}\begin{bmatrix} 2\cos(t) \\ \cos(t)-\sin(t) \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 2\sin(t) \\ \sin(t)+\cos(t) \end{bmatrix}.$$

Sub in the initial condition

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since

$$\mathbf{x}_1(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \mathbf{x}_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

we get

$$c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This gives

$$\begin{aligned} 2c_1 &= 0, \\ c_1 + c_2 &= 1. \end{aligned}$$

Thus,

$$c_1 = 0, \qquad c_2 = 1.$$

Therefore, the solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{-2t}\begin{bmatrix} 2\sin(t) \\ \sin(t)+\cos(t) \end{bmatrix}.$$

Problem 3. Solve the system $u' = 2u + 8v$, $v' = -u - 2v$, subject to the initial conditions $u(0) = 2, v(0) = -1$.

Solution 3. The system in matrix form is $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} u \\ v \end{pmatrix}$. The eigenvalues are found via $\det(A - \lambda I) = 0$:

$$\det\begin{pmatrix} 2-\lambda & 8 \\ -1 & -2-\lambda \end{pmatrix} = (2-\lambda)(-2-\lambda) + 8 = \lambda^2 - 4 + 8 = \lambda^2 + 4 = 0 \Longrightarrow \lambda = \pm 2i.$$

For $\lambda = 0 + 2i$, we have $\alpha = 0$ and $\beta = 2$. We find the eigenvector $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ by solving $(A - 2iI)\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} 2-2i & 8 \\ -1 & -2-2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow -v_1 - (2+2i)v_2 = 0 \Longrightarrow v_1 = -(2+2i)v_2.$$

Choosing $v_2 = -1$, we get $v_1 = 2 + 2i$. Thus, $\mathbf{v} = \begin{pmatrix} 2+2i \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + i\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, so $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Using the recipe for complex eigenvalues:

$$\mathbf{x}_1 = \cos(2t)\mathbf{a} - \sin(2t)\mathbf{b} = \cos(2t)\begin{pmatrix} 2 \\ -1 \end{pmatrix} - \sin(2t)\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\cos 2t - 2\sin 2t \\ -\cos 2t \end{pmatrix}$$
$$\mathbf{x}_2 = \sin(2t)\mathbf{a} + \cos(2t)\mathbf{b} = \sin(2t)\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \cos(2t)\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\sin 2t + 2\cos 2t \\ -\sin 2t \end{pmatrix}.$$

The general solution is $\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$. Applying $\mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$:

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = c_1\begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 0 \end{pmatrix} \Longrightarrow \begin{cases} 2c_1 + 2c_2 = 2 \\ -c_1 = -1 \end{cases} \Longrightarrow c_1 = 1,\ c_2 = 0.$$

The solution is $\mathbf{x}(t) = \mathbf{x}_1(t)$, which in scalar form is:

$$u(t) = 2\cos(2t) - 2\sin(2t), \quad v(t) = -\cos(2t).$$

Problem 4. Find the solution of the system

$$x' = \begin{bmatrix} 3 & 9 \\ -2 & -3 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

Solution 4. Find the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 3-\lambda & 9 \\ -2 & -3-\lambda \end{vmatrix} \\ &= (3-\lambda)(-3-\lambda) + 18 \\ &= \lambda^2 + 9. \end{aligned}$$

Thus,

$$\lambda = \pm 3i.$$

For $\lambda_1 = 3i$,

$$\begin{bmatrix} 3-3i & 9 \\ -2 & -3-3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This leads to

$$(3-3i)v_1 + 9v_2 = 0.$$

Taking $v_1 = 3$, we get

$$v_2 = -1 + i.$$

Therefore, an eigenvector is

$$V_1 = \begin{bmatrix} 3 \\ -1+i \end{bmatrix}.$$

Now write

$$e^{3it}\begin{bmatrix} 3 \\ -1+i \end{bmatrix} = (\cos(3t) + i\sin(3t))\begin{bmatrix} 3 \\ -1+i \end{bmatrix}.$$

Multiplying out,

$$e^{3it}\begin{bmatrix} 3 \\ -1+i \end{bmatrix} = \begin{bmatrix} 3\cos(3t) + 3i\sin(3t) \\ -\cos(3t) - \sin(3t) + i(\cos(3t) - \sin(3t)) \end{bmatrix}.$$

Therefore, the two real-valued solutions are

$$x_1(t) = \begin{bmatrix} 3\cos(3t) \\ -\cos(3t) - \sin(3t) \end{bmatrix},$$

and

$$x_2(t) = \begin{bmatrix} 3\sin(3t) \\ \cos(3t) - \sin(3t) \end{bmatrix}.$$

Therefore, the general solution is

$$x(t) = c_1\begin{bmatrix} 3\cos(3t) \\ -\cos(3t) - \sin(3t) \end{bmatrix} + c_2\begin{bmatrix} 3\sin(3t) \\ \cos(3t) - \sin(3t) \end{bmatrix}.$$

Sub in the initial condition

$$x(0) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}.$$

Since

$$x_1(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \qquad x_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

we get

$$c_1\begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}.$$

This gives

$$\begin{aligned} 3c_1 &= -3, \\ -c_1 + c_2 &= 0. \end{aligned}$$

Therefore,

$$c_1 = -1, \qquad c_2 = -1.$$

Thus, the solution is

$$x(t) = -\begin{bmatrix} 3\cos(3t) \\ -\cos(3t) - \sin(3t) \end{bmatrix} - \begin{bmatrix} 3\sin(3t) \\ \cos(3t) - \sin(3t) \end{bmatrix} = \begin{bmatrix} -3\cos(3t) - 3sin(3t) \\ 2\sin(3t) \end{bmatrix}$$

Problem 5. Find a real fundamental matrix of the following system of differential equations

$$
\begin{aligned}
x_1'(t) &= 2x_1(t) + 8x_2(t) \\
x_2'(t) &= -x_1(t) - 2x_2(t)
\end{aligned}
$$

Solution 5. Write the system in matrix form:

$$\mathbf{x}' = A\mathbf{x}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad A = \begin{bmatrix} 2 & 8 \\ -1 & -2 \end{bmatrix}.$$

Find the eigenvalues:

$$
\begin{aligned}
0 &= \det(A - \lambda I) \\
&= \begin{vmatrix} 2-\lambda & 8 \\ -1 & -2-\lambda \end{vmatrix} \\
&= (2-\lambda)(-2-\lambda) - (-1)(8) \\
&= -4 + \lambda^2 + 8 \\
&= \lambda^2 + 4.
\end{aligned}
$$

Thus,

$$\lambda^2 + 4 = 0 \qquad \Rightarrow \qquad \lambda = \pm 2i.$$

For $\lambda_1 = 2i$, an eigenvector $\mathbf{v}$ satisfies

$$(A - \lambda_1 I)\mathbf{v} = \mathbf{0}.$$

Therefore,

$$\begin{bmatrix} 2-2i & 8 \\ -1 & -2-2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From the first row,

$$(2-2i)a + 8b = 0.$$

Dividing by 2, we get

$$(1-i)a + 4b = 0.$$

Hence,

$$b = -\frac{1}{4}(1-i)a.$$

Taking $a = 4$, we get

$$b = -(1-i) = -1 + i.$$

Thus,

$$\mathbf{v} = \begin{bmatrix} 4 \\ -1+i \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore,

$$\mathbf{a} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since $\lambda = 0 + 2i$, the two real-valued solutions are

$$\mathbf{x}^{(1)}(t) = \mathbf{a}\cos(2t) - \mathbf{b}\sin(2t),$$

and

$$\mathbf{x}^{(2)}(t) = \mathbf{b}\cos(2t) + \mathbf{a}\sin(2t).$$

Hence,

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}\cos(2t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin(2t),$$

so

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 4\cos(2t) \\ -\cos(2t) - \sin(2t) \end{bmatrix}.$$

Also,

$$\mathbf{x}^{(2)}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos(2t) + \begin{bmatrix} 4 \\ -1 \end{bmatrix}\sin(2t),$$

so

$$\mathbf{x}^{(2)}(t) = \begin{bmatrix} 4\sin(2t) \\ \cos(2t) - \sin(2t) \end{bmatrix}.$$

Therefore, a real fundamental matrix is

$$X(t) = [\mathbf{x}^{(1)}(t) \quad \mathbf{x}^{(2)}(t)] = \begin{bmatrix} 4\cos(2t) & 4\sin(2t) \\ -\cos(2t) - \sin(2t) & \cos(2t) - \sin(2t) \end{bmatrix}.$$

Problem 6. Solve the initial value problem

$$x' = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Solution 6. Write the system as

$$\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} \mathbf{x}.$$

Find the eigenvalues:

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \begin{vmatrix} 3-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} \\ &= (3-\lambda)^2 - (-1)(1) \\ &= \lambda^2 - 6\lambda + 10. \end{aligned}$$

Thus,

$$\lambda = \frac{6 \pm \sqrt{36 - 40}}{2} = 3 \pm i.$$

For $\lambda_1 = 3 + i$,

$$(A - (3 + i)I)\mathbf{v} = \mathbf{0}.$$

Therefore,

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From row 1,

$$-ia + b = 0.$$

Hence,

$$b = ia.$$

Taking $a = 1$, we get

$$b = i.$$

Thus,

$$\mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore,

$$\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since $\lambda = 3 + i$, the two real-valued solutions are

$$\mathbf{x}_1(t) = e^{3t}(\mathbf{a}\cos(t) - \mathbf{b}\sin(t)),$$

and

$$\mathbf{x}_2(t) = e^{3t}(\mathbf{b}\cos(t) + \mathbf{a}\sin(t)).$$

Hence,

$$\mathbf{x}_1(t) = e^{3t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos(t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin(t)\right) = e^{3t}\begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}.$$

Also,

$$\mathbf{x}_2(t) = e^{3t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(t)\right) = e^{3t}\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}.$$

Therefore, the general solution is

$$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t).$$

Sub in the initial condition

$$\mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Since

$$\mathbf{x}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

we get

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence,

$$C_1 = 2, \qquad C_2 = 4.$$

Thus, the solution is

$$\mathbf{x}(t) = e^{3t}\left(2\begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + 4\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}\right).$$

Therefore,

$$\mathbf{x}(t) = e^{3t}\begin{bmatrix} 2\cos(t) + 4\sin(t) \\ 4\cos(t) - 2\sin(t) \end{bmatrix}.$$

Problem 7. Find the general solution to the differential equation system

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Here $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution 7. Find the eigenvalues of A by solving $\det(A - rI) = 0$:

$$\det\begin{bmatrix} -r & -1 \\ 1 & -r \end{bmatrix} = r^2 + 1 = 0 \Longrightarrow r = \pm i$$

We have complex eigenvalues with $\lambda = 0$ and $\mu = 1$. For $r_1 = i$, find the eigenvector $\vec{\xi}$:

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The first row gives $-i\xi_1 - \xi_2 = 0 \Longrightarrow \xi_2 = -i\xi_1$. Let $\xi_1 = i$, then $\xi_2 = -i(i) = 1$. The eigenvector is $\vec{\xi} = \begin{bmatrix} i \\ 1 \end{bmatrix}$. Decompose the complex solution into real and imaginary parts:

$$\begin{bmatrix} i \\ 1 \end{bmatrix}(\cos t + i\sin t) = \begin{bmatrix} i\cos t - \sin t \\ \cos t + i\sin t \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + i\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

Assign $\vec{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ based on the vector parts. The general solution is:

$$\mathbf{y}(t) = C_1\left(\cos t\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + C_2\left(\sin t\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \cos t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

Problem 8. Find the general solution to the differential equation system

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Solution 8. First, find the eigenvalues of A by solving the characteristic equation $\det(A - rI) = 0$:

$$\det\begin{bmatrix} 1-r & 1 \\ -1 & 1-r \end{bmatrix} = (1-r)^2 - (1)(-1) = 0$$

$$1 - 2r + r^2 + 1 = 0 \Rightarrow r^2 - 2r + 2 = 0$$

Use the quadratic formula to find the roots:

$$r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

We have complex eigenvalues $r_1 = 1 + i$ and $r_2 = 1 - i$. Here $\lambda = 1$ and $\mu = 1$.

For $r_1 = 1 + i$, find the eigenvector $\vec{u}$:

$$(A - (1+i)I)\vec{u} = \begin{bmatrix} 1-(1+i) & 1 \\ -1 & 1-(1+i) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the first row, $-iu_1 + u_2 = 0 \Rightarrow u_2 = iu_1$. Let $u_1 = 1$, then $u_2 = i$. The eigenvector is $\vec{u} = \begin{bmatrix} 1 \\ i \end{bmatrix}$.

The complex-valued solution is $e^{rt}\vec{u}$:

$$\mathbf{x}(t) = e^{(1+i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} = e^t(\cos t + i\sin t)\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= e^t\begin{bmatrix} \cos t + i\sin t \\ i\cos t - \sin t \end{bmatrix} = e^t\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + ie^t\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

Let $\vec{x}^{(1)}(t) = \text{Re}\{\mathbf{x}(t)\}$ and $\vec{x}^{(2)}(t) = \text{Im}\{\mathbf{x}(t)\}$.

$$\vec{x}^{(1)}(t) = e^t\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}, \quad \vec{x}^{(2)}(t) = e^t\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

The general solution is:

$$\mathbf{x}(t) = C_1e^t\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + C_2e^t\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

Problem 9. Find the general solution of the differential equation system

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Solution 9. First, find the eigenvalues of matrix A by solving the characteristic equation $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 0-\lambda & 1 \\ -1 & 0-\lambda \end{bmatrix} = (-\lambda)(-\lambda) - (1)(-1)$$
$$= \lambda^2 + 1 = 0$$

Solving for λ, we get purely imaginary eigenvalues:

$$\lambda^2 = -1 \Longrightarrow \lambda = \pm i$$

These are in the form $\alpha \pm \beta i$, with $\alpha = 0$ and $\beta = 1$.

Next, find the eigenvector for $\lambda_1 = i$:

$$(A - iI)\vec{v} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the first row, $-iv_1 + v_2 = 0 \Longrightarrow v_2 = iv_1$. Let $v_1 = 1$, then $v_2 = i$. The eigenvector is $\vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$.

The complex-valued solution is $e^{\lambda t}\vec{v} = e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix}$. Using Euler's formula $e^{it} = \cos t + i\sin t$:

$$e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} = (\cos t + i\sin t)\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} \cos t + i\sin t \\ i\cos t + i^2\sin t \end{bmatrix} = \begin{bmatrix} \cos t + i\sin t \\ -\sin t + i\cos t \end{bmatrix}$$

Split into real and imaginary parts:

$$\vec{x}_1(t) = \text{Re}\left\{e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix}\right\} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$$
$$\vec{x}_2(t) = \text{Im}\left\{e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix}\right\} = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

The general solution is a linear combination of these two linearly independent real solutions:

$$\vec{x}(t) = C_1\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + C_2\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

Problem 10. Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

Solution 10. First, find the eigenvalues of matrix A by solving the characteristic equation $\det(A - \lambda I) = 0$:

$$\det\begin{bmatrix}1-\lambda & 2\\ -2 & 1-\lambda\end{bmatrix} = (1-\lambda)(1-\lambda)-(2)(-2)=0$$

$$(1-\lambda)^2+4=0 \Longrightarrow 1-2\lambda+\lambda^2+4=0 \Longrightarrow \lambda^2-2\lambda+5=0$$

Use the quadratic formula to find the eigenvalues:

$$\lambda = \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(5)}}{2(1)} = \frac{2\pm\sqrt{4-20}}{2} = \frac{2\pm\sqrt{-16}}{2} = \frac{2\pm 4i}{2} = 1\pm 2i$$

We have complex eigenvalues $\lambda_1 = 1+2i$ and $\lambda_2 = 1-2i$. Here, $\alpha = 1$ and $\beta = 2$.

Next, find the eigenvector $\vec{k}^{(1)}$ corresponding to the eigenvalue $\lambda_1 = 1+2i$. We solve $(A-(1+2i)I)\vec{k}^{(1)} = \mathbf{0}$:

$$\begin{bmatrix}1-(1+2i) & 2\\ -2 & 1-(1+2i)\end{bmatrix}\begin{bmatrix}k_1\\ k_2\end{bmatrix} = \begin{bmatrix}-2i & 2\\ -2 & -2i\end{bmatrix}\begin{bmatrix}k_1\\ k_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

From the first row, $-2ik_1+2k_2=0 \Longrightarrow k_2 = ik_1$. Let $k_1 = 1$, then $k_2 = i$. The eigenvector is $\vec{k}^{(1)} = \begin{bmatrix}1\\ i\end{bmatrix}$.

The complex-valued solution is $e^{\lambda_1 t}\vec{k}^{(1)} = e^{(1+2i)t}\begin{bmatrix}1\\ i\end{bmatrix}$. Using Euler's formula $e^{(1+2i)t} = e^t(\cos(2t)+i\sin(2t))$:

$$e^t(\cos(2t)+i\sin(2t))\begin{bmatrix}1\\ i\end{bmatrix} = e^t\begin{bmatrix}\cos(2t)+i\sin(2t)\\ i\cos(2t)+i^2\sin(2t)\end{bmatrix} = e^t\begin{bmatrix}\cos(2t)+i\sin(2t)\\ -\sin(2t)+i\cos(2t)\end{bmatrix}$$

Separate this into its real and imaginary parts:

$$\vec{x}^{(1)}(t) = \text{Re}\left\{e^{(1+2i)t}\begin{bmatrix}1\\ i\end{bmatrix}\right\} = e^t\begin{bmatrix}\cos(2t)\\ -\sin(2t)\end{bmatrix}$$

$$\vec{x}^{(2)}(t) = \text{Im}\left\{e^{(1+2i)t}\begin{bmatrix}1\\ i\end{bmatrix}\right\} = e^t\begin{bmatrix}\sin(2t)\\ \cos(2t)\end{bmatrix}$$

The general solution is a linear combination of these two linearly independent real solutions:

$$\mathbf{x}(t) = C_1e^t\begin{bmatrix}\cos(2t)\\ -\sin(2t)\end{bmatrix} + C_2e^t\begin{bmatrix}\sin(2t)\\ \cos(2t)\end{bmatrix}$$

Problem 11. Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$.

Solution 11. First, find the eigenvalues by solving $\det(A-\lambda I) = 0$:

$$\det\begin{pmatrix}1-\lambda & 1\\ -1 & 1-\lambda\end{pmatrix} = (1-\lambda)^2-(1)(-1) = 0$$

$$(1-\lambda)^2+1=0 \Longrightarrow \lambda^2-2\lambda+1+1=0 \Longrightarrow \lambda^2-2\lambda+2=0$$

Using the quadratic formula:

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

For $\lambda_1 = 1 + i$, solve $(A - \lambda_1 I)\vec{v} = \vec{0}$:

$$\begin{pmatrix} 1-(1+i) & 1 \\ -1 & 1-(1+i) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From the first row: $-ia + b = 0 \Longrightarrow b = ia$. Let $a = 1$, then $b = i$. The eigenvector is $\vec{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{v}_r + i\vec{v}_i$. The real and imaginary parts of $e^{\lambda_1 t}\vec{v}_1$ form two linearly independent real solutions:

$$\mathbf{x}_1(t) = e^t(\vec{v}_r \cos t - \vec{v}_i \sin t) = e^t\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t\right) = e^t\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

$$\mathbf{x}_2(t) = e^t(\vec{v}_r \sin t + \vec{v}_i \cos t) = e^t\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t\right) = e^t\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

The general solution is a linear combination of these two solutions:

$$\mathbf{x}(t) = C_1 e^t\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2 e^t\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

9.7 - Nonhomogeneous Systems

Problem 12. Given that $\mathbf{x}(t) = c_1\begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} + c_2\begin{pmatrix} 2e^{-2t} \\ 3e^{-2t} \end{pmatrix}$ is the general solution for the homogeneous system $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\mathbf{x}$, solve the nonhomogeneous initial value problem $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}e^{-t}$, $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

Solution 12. The homogeneous solution is $\mathbf{x}_h(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{-2t}$. The nonhomogeneous term is $\mathbf{g}(t) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}e^{-t}$. Since e^{-t} is a solution to the homogeneous system, we assume a particular solution of the form

$$\mathbf{x}_p(t) = \mathbf{a}te^{-t} + \mathbf{b}e^{-t}.$$

Substituting $\mathbf{x}_p$ into $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$ gives

$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} = A\mathbf{a}te^{-t} + A\mathbf{b}e^{-t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}e^{-t}.$$

Equating the coefficients of te^{-t}:

$$-\mathbf{a} = A\mathbf{a} \Longrightarrow (A+I)\mathbf{a} = \mathbf{0} \Longrightarrow \begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow a_1 = a_2.$$

Equating the coefficients of e^{-t}:

$$\mathbf{a} - \mathbf{b} = A\mathbf{b} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} \Longrightarrow (A+I)\mathbf{b} = \mathbf{a} - \begin{pmatrix} 2 \\ 2 \end{pmatrix} \Longrightarrow \begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 - 2 \\ a_1 - 2 \end{pmatrix}.$$

For this system to be consistent, the rows must be linearly dependent, which is already satisfied for any a_1. We choose $a_1 = 2$ to simplify the equation for $\mathbf{b}$, giving $\mathbf{a} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$. Then $(A+I)\mathbf{b} = \mathbf{0}$, so we can set $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus, the particular solution is $\mathbf{x}_p(t) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{-t}$. The general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{-t}.$$

Applying the initial condition $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$:

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Longrightarrow \begin{cases} c_1 + 2c_2 = 3 \\ c_1 + 3c_2 = 5 \end{cases}.$$

Subtracting the first equation from the second gives $c_2 = 2$. Substituting back gives $c_1 = -1$. The final solution is

$$\mathbf{x}(t) = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + 2\begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{-t} = \begin{pmatrix} -e^{-t} + 4e^{-2t} + 2te^{-t} \\ -e^{-t} + 6e^{-2t} + 2te^{-t} \end{pmatrix}.$$

Problem 13. Use the method of variation of parameters for systems to solve the system

$$\begin{cases} \dfrac{dx}{dt} = -y + \sec(t) \\ \dfrac{dy}{dt} = x \end{cases}$$

given that the general solution to the associated homogeneous system is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} + c_2 \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}.$$

Solution 13. The system is $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t)$ where $\mathbf{f}(t) = \begin{pmatrix} \sec(t) \\ 0 \end{pmatrix}$. A fundamental matrix $X(t)$ of the associated homogeneous equation is

$$X(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$

The determinant is $\det(X(t)) = \cos^2(t) + \sin^2(t) = 1$. The inverse matrix is

$$X^{-1}(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

A particular solution is $\mathbf{x}_p(t) = X(t)\mathbf{u}(t)$ where

$$\mathbf{u}(t) = \int X^{-1}(t)\mathbf{f}(t)dt = \int \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} \sec(t) \\ 0 \end{pmatrix} dt.$$

Computing the product inside the integral:

$$X^{-1}(t)\mathbf{f}(t) = \begin{pmatrix} \cos(t)\sec(t) \\ -\sin(t)\sec(t) \end{pmatrix} = \begin{pmatrix} 1 \\ -\tan(t) \end{pmatrix}.$$

Integrating each component:

$$\mathbf{u}(t) = \int \begin{pmatrix} 1 \\ -\tan(t) \end{pmatrix} dt = \begin{pmatrix} t \\ \ln|\cos(t)| \end{pmatrix}.$$

The particular solution is

$$\mathbf{x}_p(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} t \\ \ln|\cos(t)| \end{pmatrix} = \begin{pmatrix} t\cos(t) - \sin(t)\ln|\cos(t)| \\ t\sin(t) + \cos(t)\ln|\cos(t)| \end{pmatrix}.$$

The general solution is $\mathbf{x}(t) = X(t)\mathbf{c} + \mathbf{x}_p(t)$:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} + c_2 \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} + \begin{pmatrix} t\cos(t) - \sin(t)\ln|\cos(t)| \\ t\sin(t) + \cos(t)\ln|\cos(t)| \end{pmatrix}.$$

Problem 14. Given that $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}$ and $\mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$ form a fundamental set of solutions for $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$, solve the initial value problem $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} te^{2t} \\ te^{2t} \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$.

Solution 14. The nonhomogeneous term is $\mathbf{f}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{2t}$. Since $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$ is a homogeneous solution, we use the method of undetermined coefficients with the guess:

$$\mathbf{x}_p(t) = \mathbf{a}t^2e^{2t} + \mathbf{b}te^{2t} + \mathbf{d}e^{2t}.$$

Alternatively, using variation of parameters $\mathbf{x}_p(t) = X(t)\mathbf{u}(t)$, where $X(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$:

$$\det(X(t)) = e^{-3t}e^{2t} - (-4e^{-3t}e^{2t}) = 5e^{-t}.$$

The inverse is $X^{-1}(t) = \frac{1}{5e^{-t}} \begin{pmatrix} e^{2t} & -e^{2t} \\ 4e^{-3t} & e^{-3t} \end{pmatrix} = \frac{1}{5} \begin{pmatrix} e^{3t} & -e^{3t} \\ 4e^{-2t} & e^{-2t} \end{pmatrix}$. Compute $\mathbf{u}'(t) = X^{-1}(t)\mathbf{f}(t)$:

$$\mathbf{u}'(t) = \frac{1}{5} \begin{pmatrix} e^{3t} & -e^{3t} \\ 4e^{-2t} & e^{-2t} \end{pmatrix} \begin{pmatrix} te^{2t} \\ te^{2t} \end{pmatrix} = \frac{1}{5} \begin{pmatrix} te^{5t} - te^{5t} \\ 4t + t \end{pmatrix} = \begin{pmatrix} 0 \\ t \end{pmatrix}.$$

Integrating gives $\mathbf{u}(t) = \int \begin{pmatrix} 0 \\ t \end{pmatrix} dt = \begin{pmatrix} 0 \\ \frac{1}{2}t^2 \end{pmatrix}$. Thus, the particular solution is:

$$\mathbf{x}_p(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{2}t^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}t^2 e^{2t} \\ \frac{1}{2}t^2 e^{2t} \end{pmatrix}.$$

The general solution is $\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{2}t^2 e^{2t}$. Using the initial condition $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$:

$$\begin{pmatrix} 0 \\ 5 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Longrightarrow \begin{cases} c_1 + c_2 = 0 \\ -4c_1 + c_2 = 5 \end{cases}.$$

From the first equation, $c_2 = -c_1$. Substituting into the second: $-4c_1 - c_1 = 5 \Longrightarrow c_1 = -1$, so $c_2 = 1$. The final solution to the initial value problem is:

$$\mathbf{x}(t) = -\begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{2}t^2 e^{2t}.$$

Problem 15. Find the general solution of the system $x' = 2x + 3y - 7$, $y' = -x - 2y + 5$.

Solution 15. The system in matrix form is $\mathbf{x}' = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -7 \\ 5 \end{pmatrix}$. The eigenvalues of $A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$ are found via $\det(A - \lambda I) = \lambda^2 - 1 = 0$, so $\lambda = \pm 1$. For $\lambda_1 = 1$, $(A - I)\mathbf{v} = \mathbf{0} \Longrightarrow \begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \mathbf{v}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$. For $\lambda_2 = -1$, $(A + I)\mathbf{v} = \mathbf{0} \Longrightarrow \begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The homogeneous solution is $\mathbf{x}_h(t) = c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$. For the particular solution $\mathbf{x}_p = \begin{pmatrix} a \\ b \end{pmatrix}$, we substitute into the system:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} -7 \\ 5 \end{pmatrix} \Longrightarrow \begin{cases} 2a + 3b = 7 \\ -a - 2b = -5 \end{cases} \Longrightarrow a = -1,\ b = 3.$$

The general solution is $\mathbf{x}(t) = c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, or equivalently,

$$\begin{aligned} x(t) &= -3c_1 e^t + c_2 e^{-t} - 1 \\ y(t) &= c_1 e^t - c_2 e^{-t} + 3 \end{aligned}$$

Problem 16. Let $A = \begin{pmatrix} -1/10 & 3/40 \\ 1/10 & -1/5 \end{pmatrix}$. Given that $\Psi(t) = \begin{pmatrix} 3e^{-t/20} & -e^{-t/4} \\ 2e^{-t/20} & 2e^{-t/4} \end{pmatrix}$ is a fundamental matrix for the homogeneous system $\mathbf{x}' = A\mathbf{x}$, find the general solution of the nonhomogeneous system $\mathbf{x}' = A\mathbf{x} + \begin{pmatrix} 6 \\ 12 \end{pmatrix}$.

Solution 16. The homogeneous solution is $\mathbf{x}_h(t) = \Psi(t)\mathbf{c} = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t/20} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t/4}$. Since the nonhomogeneous term is a constant vector $\mathbf{g} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$ and $\lambda = 0$ is not an eigenvalue of A, we assume a particular solution of the form $\mathbf{x}_p = \begin{pmatrix} a \\ b \end{pmatrix}$. Substituting into $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$ yields:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/10 & 3/40 \\ 1/10 & -1/5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 6 \\ 12 \end{pmatrix} \Rightarrow \begin{pmatrix} -1/10 & 3/40 \\ 1/10 & -1/5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -6 \\ -12 \end{pmatrix}.$$

This gives the system of equations: 1) $-\frac{1}{10}a + \frac{3}{40}b = -6 \Rightarrow -4a + 3b = -240$ 2) $\frac{1}{10}a - \frac{1}{5}b = -12 \Rightarrow a - 2b = -120 \Rightarrow a = 2b - 120$ Substitute (2) into (1):

$$-4(2b - 120) + 3b = -240 \Rightarrow -8b + 480 + 3b = -240 \Rightarrow -5b = -720 \Rightarrow b = 144.$$

Then, $a = 2(144) - 120 = 288 - 120 = 168$. Thus, the particular solution is $\mathbf{x}_p = \begin{pmatrix} 168 \\ 144 \end{pmatrix}$. The general solution is:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t/20} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t/4} + \begin{pmatrix} 168 \\ 144 \end{pmatrix}.$$

Problem 17. Find the general solution of $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Solution 17. First, solve the homogeneous system $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The eigenvalues are found via $\det(A - \lambda I) = 0$:

$$\det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i.$$

For $\lambda = i$ $(\alpha = 0, \beta = 1)$, we find the eigenvector $\mathbf{v} = \mathbf{a} + i\mathbf{b}$:

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -iv_1 + v_2 = 0.$$

Choosing $v_1 = 1$, we get $v_2 = i$. Thus, $\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The homogeneous solution is:

$$\mathbf{x}_h(t) = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

For the particular solution, since the nonhomogeneous term is a constant vector $\mathbf{g} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we guess $\mathbf{x}_p = \begin{pmatrix} a \\ b \end{pmatrix}$:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} b + 1 \\ -a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives $b = -1$ and $a = 0$, so $\mathbf{x}_p = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. The general solution is:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Problem 18. Let $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ and $\mathrm{f}(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$. Given that the fundamental matrix for the system $\mathrm{x}' = A\mathrm{x}$ is $X(t) = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}$, find the general solution of the nonhomogeneous system $\mathrm{x}' = A\mathrm{x} + \mathrm{f}(t)$

Solution 18. For a nonhomogeneous system

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t),$$

the general solution is

$$\mathbf{x}(t) = X(t)\mathbf{c} + X(t) \int X(t)^{-1} \mathbf{f}(t)\, dt.$$

We are given

$$X(t) = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}, \qquad \mathbf{f}(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}.$$

First compute $X(t)^{-1}$. Since

$$\det(X(t)) = 3 - 1 = 2,$$

we have

$$X(t)^{-1} = \frac{1}{2} \begin{bmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{bmatrix}.$$

Now compute

$$X(t)^{-1}\mathbf{f}(t) = \frac{1}{2} \begin{bmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{bmatrix} \begin{bmatrix} e^t \\ -e^t \end{bmatrix}.$$

Thus,

$$\begin{aligned} X(t)^{-1}\mathbf{f}(t) &= \frac{1}{2} \begin{bmatrix} 3e^{-t}e^t + e^{-t}e^t \\ -e^te^t - e^te^t \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 4 \\ -2e^{2t} \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -e^{2t} \end{bmatrix}. \end{aligned}$$

Integrate:

$$\int X(t)^{-1}\mathbf{f}(t)\, dt = \int \begin{bmatrix} 2 \\ -e^{2t} \end{bmatrix} dt = \begin{bmatrix} 2t \\ -\frac{1}{2}e^{2t} \end{bmatrix}.$$

Therefore, a particular solution is

$$\mathbf{x}_p(t) = X(t)\begin{bmatrix} 2t \\ -\frac{1}{2}e^{2t} \end{bmatrix}.$$

Hence,

$$\begin{aligned}\mathbf{x}_p(t) &= \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}\begin{bmatrix} 2t \\ -\frac{1}{2}e^{2t} \end{bmatrix} \\ &= \begin{bmatrix} 2te^t - \frac{1}{2}e^t \\ 2te^t - \frac{3}{2}e^t \end{bmatrix}.\end{aligned}$$

Thus, the general solution is

$$\mathbf{x}(t) = c_1\begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2\begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix} + \begin{bmatrix} 2te^t - \frac{1}{2}e^t \\ 2te^t - \frac{3}{2}e^t \end{bmatrix}.$$

Problem 19. Given that $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$ is a solution of the homogeneous system $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\mathbf{x}$, find the general solution of the nonhomogeneous differential equation system

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Solution 19. Write the system as

$$\mathbf{x}' = A\mathbf{x} + \mathbf{b},$$

where

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

We are given one solution of the homogeneous system:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$$

So $\lambda_1 = 2$. Find the other eigenvalue using

$$\det(A - \lambda I) = 0.$$

$$\begin{aligned}\det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} \\ &= (1-\lambda)(3-\lambda) + 1 \\ &= \lambda^2 - 4\lambda + 4 \\ &= (\lambda - 2)^2.\end{aligned}$$

Thus, $\lambda = 2$ is repeated.

Find a generalized eigenvector $\mathbf{w}$ satisfying

$$(A - 2I)\mathbf{w} = \mathbf{v},$$

where

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Since

$$A - 2I = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix},$$

solve

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This gives

$$-w_1 - w_2 = 1.$$

Taking $w_1 = -1$, we get $w_2 = 0$. So

$$\mathbf{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Therefore, the homogeneous solution is

$$\mathbf{x}_h(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t}\left[t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right].$$

Now find a constant particular solution $\mathbf{x}_p$. Since $\mathbf{x}_p' = 0$, we need

$$A\mathbf{x}_p + \mathbf{b} = 0.$$

Let

$$\mathbf{x}_p = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Then

$$\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$a - b + 1 = 0,$$
$$a + 3b - 3 = 0.$$

Solving,

$$a = 0, \qquad b = 1.$$

Therefore,

$$\mathbf{x}_p = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, the general solution is

$$\mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \left[t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Problem 20 (25). Find a particular solution of the nonhomogeneous system

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ -e^{-t} \end{pmatrix}.$$

You may use that

$$\mathbf{x} = C_1 \begin{pmatrix} e^{t} \\ -3e^{t} \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$$

is the general solution of the associated homogeneous system.

Solution 20. We look for a particular solution of the form

$$\mathbf{x}_p(t) = \begin{pmatrix} ae^{-t} \\ be^{-t} \end{pmatrix}.$$

Then

$$\mathbf{x}_p'(t) = \begin{pmatrix} -ae^{-t} \\ -be^{-t} \end{pmatrix}.$$

Substitute into the system:

$$\begin{pmatrix} -ae^{-t} \\ -be^{-t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} ae^{-t} \\ be^{-t} \end{pmatrix} + \begin{pmatrix} 2e^{-t} \\ -e^{-t} \end{pmatrix}.$$

This gives

$$-ae^{-t} = ae^{-t} + 2e^{-t},$$
$$-be^{-t} = 3ae^{-t} + 2be^{-t} - e^{-t}.$$

Divide by e^{-t}:

$$-a = a + 2,$$
$$-b = 3a + 2b - 1.$$

From the first equation,

$$a = -1.$$

Substitute $a = -1$ into the second equation:

$$
\begin{aligned}
-b &= 3(-1) + 2b - 1 \\
-b &= 2b - 4 \\
-3b &= -4 \\
b &= \frac{4}{3}.
\end{aligned}
$$

Therefore, a particular solution is

$$\mathbf{x}_p(t) = \begin{pmatrix} -e^{-t} \\ \frac{4}{3}e^{-t} \end{pmatrix}.$$

Problem 21. Find the general solution of the nonhomogeneous system

$$x' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} e^{t} \\ e^{-t} \end{bmatrix}.$$

The fundamental matrix for the associated homogeneous system is given by:

$$\begin{bmatrix} e^{-t} & e^{3t} \\ -e^{-t} & 3e^{3t} \end{bmatrix}$$

Solution 21. For a nonhomogeneous system

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t),$$

the general solution is

$$\mathbf{x}(t) = X(t)\mathbf{c} + X(t)\int X(t)^{-1}\mathbf{f}(t)\,dt.$$

We are given

$$X(t) = \begin{bmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{bmatrix}, \qquad \mathbf{f}(t) = \begin{bmatrix} e^{t} \\ e^{-t} \end{bmatrix}.$$

First compute $X(t)^{-1}$. Since

$$\det(X(t)) = 2e^{2t},$$

we have

$$X(t)^{-1} = \frac{1}{2e^{2t}} \begin{bmatrix} e^{3t} & -e^{3t} \\ e^{-t} & e^{-t} \end{bmatrix}.$$

Now compute

$$X(t)^{-1}\mathbf{f}(t) = \frac{1}{2e^{2t}} \begin{bmatrix} e^{3t} & -e^{3t} \\ e^{-t} & e^{-t} \end{bmatrix} \begin{bmatrix} e^{t} \\ e^{-t} \end{bmatrix}.$$

Thus,

$$X(t)^{-1}\mathbf{f}(t) = \frac{1}{2e^{2t}}\begin{bmatrix} e^{4t} - e^{2t} \\ 1 + e^{-2t} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2}e^{2t} - \frac{1}{2} \\ \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-4t} \end{bmatrix}.$$

Integrate:

$$\int X(t)^{-1}\mathbf{f}(t)\,dt = \begin{bmatrix} \frac{1}{4}e^{2t} - \frac{1}{2}t \\ -\frac{1}{4}e^{-2t} - \frac{1}{8}e^{-4t} \end{bmatrix}.$$

Therefore, a particular solution is

$$\mathbf{x}_p(t) = X(t)\begin{bmatrix} \frac{1}{4}e^{2t} - \frac{1}{2}t \\ -\frac{1}{4}e^{-2t} - \frac{1}{8}e^{-4t} \end{bmatrix}.$$

Hence,

$$\mathbf{x}_p(t) = \begin{bmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{bmatrix}\begin{bmatrix} \frac{1}{4}e^{2t} - \frac{1}{2}t \\ -\frac{1}{4}e^{-2t} - \frac{1}{8}e^{-4t} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{4}e^{t} - \frac{1}{2}te^{-t} - \frac{1}{4}e^{t} - \frac{1}{8}e^{-t} \\ -\frac{1}{4}e^{t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{t} - \frac{1}{8}e^{-t} \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{1}{2}te^{-t} - \frac{1}{8}e^{-t} \\ -\frac{1}{2}e^{t} + \frac{1}{2}te^{-t} - \frac{1}{8}e^{-t} \end{bmatrix}.$$

Thus, the general solution is

$$\mathbf{x}(t) = c_1\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} + c_2\begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}te^{-t} - \frac{1}{8}e^{-t} \\ -\frac{1}{2}e^{t} + \frac{1}{2}te^{-t} - \frac{1}{8}e^{-t} \end{bmatrix}.$$

Problem 22. Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{t}.$$

You may use that

$$X(t) = \begin{pmatrix} e^{2t} & e^{3t} \\ -e^{2t} & -2e^{3t} \end{pmatrix}$$

is a fundamental matrix of the associated homogeneous system.

Solution 22. The general solution is of the form

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t), \qquad \mathbf{x}_h(t) = X(t)\mathbf{c}.$$

To find a particular solution, use the method of undetermined coefficients. Try

$$\mathbf{x}_p(t) = \mathbf{a}e^t, \qquad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Then

$$\mathbf{x}_p'(t) = \mathbf{a}e^t.$$

Substitute into the differential equation:

$$\mathbf{a}e^t = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \mathbf{a}e^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t.$$

Divide by e^t:

$$\mathbf{a} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \mathbf{a} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Rearranging,

$$\left(\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \mathbf{a} = -\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix} \mathbf{a} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

This gives

$$\begin{aligned} -a_2 &= -1, \\ 2a_1 + 3a_2 &= 1. \end{aligned}$$

Hence,

$$a_2 = 1, \qquad 2a_1 + 3 = 1 \Rightarrow 2a_1 = -2 \Rightarrow a_1 = -1.$$

Therefore,

$$\mathbf{a} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_p(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^t.$$

Thus, the general solution is

$$\mathbf{x}(t) = \begin{pmatrix} e^{2t} & e^{3t} \\ -e^{2t} & -2e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{t}.$$

Problem 23. Use Variation of Parameters to find a particular solution to the system

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} = A\mathbf{x} + \mathbf{f}$$

given that the corresponding homogeneous system produces a fundamental matrix

$$X(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix}$$

Solution 23. First, find the inverse fundamental matrix $X^{-1}(t)$:

$$\det(X) = e^{3t}(-e^{-t}) - e^{-t}(e^{3t}) = -e^{2t} - e^{2t} = -2e^{2t}$$

$$X^{-1} = \frac{1}{-2e^{2t}} \begin{bmatrix} -e^{-t} & -e^{-t} \\ -e^{3t} & e^{3t} \end{bmatrix} = \frac{1}{2e^{2t}} \begin{bmatrix} e^{-t} & e^{-t} \\ e^{3t} & -e^{3t} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-3t} \\ \frac{1}{2}e^{t} & -\frac{1}{2}e^{t} \end{bmatrix}$$

Now, calculate $X^{-1}\mathbf{f}$:

$$X^{-1}\mathbf{f} = \begin{bmatrix} \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-3t} \\ \frac{1}{2}e^{t} & -\frac{1}{2}e^{t} \end{bmatrix} \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} \\ \frac{1}{2}e^{4t} - \frac{1}{2}e^{4t} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Integrate:

$$\mathbf{v} = \int \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Finally, $\mathbf{x}_p = X(t)\mathbf{v}$:

$$\mathbf{x}_p = \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix} \begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} te^{3t} \\ te^{3t} \end{bmatrix}$$

Problem 24. Find a particular solution of the nonhomogeneous system

$$\mathbf{x}' = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t},$$

where

$$\Phi(t) = \begin{bmatrix} 2e^{t} & e^{2t} \\ e^{t} & e^{2t} \end{bmatrix}$$

is a fundamental matrix of the associated homogeneous system.

Solution 24. The formula for the particular solution using variation of parameters is:

$$\mathbf{x}_p(t) = \Phi(t) \int \Phi^{-1}(t)\mathbf{F}(t)dt$$

Here $\mathbf{F}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$.

Step 1: Calculate the inverse of the fundamental matrix $\Phi^{-1}(t)$. The determinant of $\Phi(t)$ is:

$$\det(\Phi) = (2e^t)(e^{2t}) - (e^{2t})(e^t) = 2e^{3t} - e^{3t} = e^{3t}$$

The inverse matrix is:

$$\Phi^{-1}(t) = \frac{1}{e^{3t}} \begin{bmatrix} e^{2t} & -e^{2t} \\ -e^t & 2e^t \end{bmatrix} = \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix}$$

Step 2: Calculate the product $\Phi^{-1}(t)\mathbf{F}(t)$.

$$\Phi^{-1}(t)\mathbf{F}(t) = \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix} \begin{bmatrix} e^t \\ -e^t \end{bmatrix} = \begin{bmatrix} (e^{-t})(e^t) + (-e^{-t})(-e^t) \\ (-e^{-2t})(e^t) + (2e^{-2t})(-e^t) \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 \\ -e^{-t} - 2e^{-t} \end{bmatrix} = \begin{bmatrix} 2 \\ -3e^{-t} \end{bmatrix}$$

Step 3: Integrate the result from Step 2, $\int \Phi^{-1}(t)\mathbf{F}(t)dt$.

$$\int \begin{bmatrix} 2 \\ -3e^{-t} \end{bmatrix} dt = \begin{bmatrix} \int 2\,dt \\ \int -3e^{-t}\,dt \end{bmatrix} = \begin{bmatrix} 2t \\ 3e^{-t} \end{bmatrix}$$

Step 4: Calculate the particular solution $\mathbf{x}_p(t) = \Phi(t) \int \Phi^{-1}(t)\mathbf{F}(t)dt$.

$$\mathbf{x}_p(t) = \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix} \begin{bmatrix} 2t \\ 3e^{-t} \end{bmatrix} = \begin{bmatrix} (2e^t)(2t) + (e^{2t})(3e^{-t}) \\ (e^t)(2t) + (e^{2t})(3e^{-t}) \end{bmatrix}$$

$$= \begin{bmatrix} 4te^t + 3e^t \\ 2te^t + 3e^t \end{bmatrix} = e^t \begin{bmatrix} 4t+3 \\ 2t+3 \end{bmatrix}$$

Problem 25. Given that $\Psi(t) = \begin{pmatrix} e^{2t} & e^{3t} \\ -e^{2t} & -2e^{3t} \end{pmatrix}$ is the fundamental matrix of the homogeneous system $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \mathbf{x}$, find the general solution of:

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

Solution 25. The general solution is $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$, where $\mathbf{x}_h = \Psi(t)\vec{C}$. For $\mathbf{x}_p$, the forcing term is $\vec{f}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$. We use the Method of Undetermined Coefficients. Assume a particular solution of the form $\mathbf{x}_p = \vec{a}e^t$, where $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$. Then $\mathbf{x}_p' = \vec{a}e^t$. Substitute into the non-homogeneous differential equation:

$$\vec{a}e^t = A(\vec{a}e^t) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$$

Divide by e^t:

$$\vec{a} = A\vec{a} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Rearrange to solve for $\vec{a}$:

$$(A - I)\vec{a} = -\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Substitute matrix $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$:

$$\left(\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

From the first row: $0a_1 - 1a_2 = -1 \Longrightarrow -a_2 = -1 \Longrightarrow a_2 = 1$. From the second row: $2a_1 + 3a_2 = 1$. Substitute $a_2 = 1$:

$$2a_1 + 3(1) = 1 \Longrightarrow 2a_1 = -2 \Longrightarrow a_1 = -1$$

So, the particular solution is:

$$\mathbf{x}_p = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^t$$

The general solution is the sum of the homogeneous and particular solutions:

$$\mathbf{x}(t) = \Psi(t)\vec{C} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^t = C_1\begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} + C_2\begin{pmatrix} e^{3t} \\ -2e^{3t} \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^t$$