Functional Analysis

Lecture Notes

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Version from December 3, 1999

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CHAPTER 1

Banach Spaces

1.1. Normed Linear Spaces

DEFINITION 1.1. If $\mathcal{X}$ is a vector space over $\mathbb{F}$, a *norm* is a function $\|\cdot\| : \mathcal{X} \to [0, \infty)$ having the properties:

(a) $x = 0$ if $\|x\| = 0$;
(b) $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{F}$ and $x \in \mathcal{X}$;
(c) $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in \mathcal{X}$.

A *normed linear space* (or short: *normed space*) is a pair $(\mathcal{X}, \|\cdot\|)$, where $\mathcal{X}$ is a vector space and $\|\cdot\|$ is a norm on $\mathcal{X}$.

LEMMA 1.2. *If $\mathcal{X}$ is a normed space, then*

(a) $\|-x\| = \|x\|$ *for all* $x \in \mathcal{X}$*;*
(b) $|\|x\| - \|y\|| \le \|x - y\|$ *for all* $x, y \in \mathcal{X}$*.*

EXAMPLE 1.3. (a) $C[a, b]$ with $\|x\| = \|x\|_\infty = \max_{t\in[a,b]} |x(t)|$;
(b) $C^1[a, b]$ with $\|x\| = \max_{t\in[a,b]} |x(t)| + \max_{t\in[a,b]} |\dot{x}(t)|$;
(c) $C[a, b]$ with $\|x\| = \int_a^b |x(t)|\mathrm{d}t$;
(d) $BV[a, b]$ with $\|x\| = |x(a)| + V_a^b(x)$.

DEFINITION 1.4. A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ that satisfies for all $x, y, z \in X$ all of the following:

(a) $d(x, y) \ge 0$ and $d(x, y) = 0$ iff $x = y$;
(b) $d(x, y) = d(y, x)$;
(c) $d(x, y) \le d(x, z) + d(z, y)$.

The pair (X, d) is then called a *metric space.*

EXAMPLE 1.5. The following are all metrics in $\mathbb{R}^2$:

(a) euclidean metric;
(b) American city metric;
(c) French railroad metric.

LEMMA 1.6. *If $(\mathcal{X}, \|\cdot\|)$ is a normed space, then $(\mathcal{X}, d)$ with d defined by $d(x, y) = \|x - y\|$ is a metric space.*

DEFINITION 1.7. Let (X, d) be a metric space. The sequence $\{x_n\}$ *converges* to x provided $d(x_n, x) \to 0$ as $n \to \infty$. We write $x_n \to x$.

LEMMA 1.8. *Let $\{x_n\}$ be a sequence in a normed space $(X, \|\cdot\|)$.*

(a) *If $x_n \to x$, then $\|x_n\| \to \|x\|$ as $n \to \infty$;*
(b) *If $\{x_n\}$ converges, then the limit is unique.*

DEFINITION 1.9. Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be normed spaces. A mapping $T : \mathcal{X} \to \mathcal{Y}$ is called *continuous* at $x_0 \in \mathcal{X}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - x_0\|_{\mathcal{X}} < \delta$ implies $\|T(x) - T(x_0)\|_{\mathcal{Y}} < \varepsilon$.

LEMMA 1.10. *A mapping T from a normed space $\mathcal{X}$ into a normed space $\mathcal{Y}$ is continuous at $x_0 \in \mathcal{X}$ if and only if $x_n \to x_0$ implies $T(x_n) \to T(x_0)$ as $n \to \infty$.*

DEFINITION 1.11. Let (X, d) be a metric space. The sequence $\{x_n\}$ is said to to be *Cauchy* provided $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

LEMMA 1.12. *Consider a normed space.*

(a) *Every convergent sequence is Cauchy.*
(b) *Every Cauchy sequence is bounded.*

DEFINITION 1.13. A metric space is called *complete* if every Cauchy sequence in it has a limit in it. A complete normed linear space is called a *Banach space.*

EXAMPLE 1.14. (a) $(\mathbb{R}, |\cdot|)$ is a Banach space.

(b) Consider $C[0,1]$ with $\|x\| = \int_0^1 |x(t)| \mathrm{d}t$. The sequence $\{x_n\}$ in $C[0,1]$ defined by

$$x_n(t) = \begin{cases} 0 & \text{for } 0 \le t \le \frac{1}{2} - \frac{1}{n} \\ nt - \frac{n}{2} + 1 & \text{for } \frac{1}{2} - \frac{1}{n} \le t \le \frac{1}{2} \\ 1 & \text{for } t \ge \frac{1}{2} \end{cases}$$

is Cauchy because of $\|x_n - x_m\| = \frac{1}{2}|\frac{1}{n} - \frac{1}{m}|$, but it is not convergent.

(c) Let $l_0 = \{x = \{\xi_n\} :$ there exists N with $\xi_n = 0$ for all $n \ge N\}$, and define a norm by $\|x\| = \max_{n \in \mathbb{N}} |\xi_n|$. Define a sequence $\{x_n\}$ in l_0 by

$$x_n = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots\right\}.$$

The sequence is Cauchy because of $\|x_n - x_m\| = \max\{\frac{1}{n+1}, \frac{1}{m+1}\}$, but it is not convergent.

(d) $(C[0,1], \|\cdot\|_\infty)$ is a Banach space.

(e) For $1 \leq p < \infty$, $(l^p, \|\cdot\|_p)$ is a Banach space, where l^p consists of all sequences $x = (\xi_k)$ with $\|x\|_p = \left\{\sum_{k\in\mathbb{N}} |\xi_k|^p\right\}^{\frac{1}{p}} < \infty$.

(f) $(l^\infty, \|\cdot\|_\infty)$ is a Banach space, where l^∞ consists of all sequences $x = (\xi_k)$ with $\|x\|_\infty = \sup_{k\in\mathbb{N}} |\xi_k| < \infty$.

DEFINITION 1.15. Let (X, d) be a metric space and suppose $P \subset X$.

(a) The point $p \in P$ is called an *interior point* of P if there is an $\varepsilon > 0$ such that all $x \in X$ with $d(x,p) < \varepsilon$ are elements of P. The collection of all interior points of P is denoted by $\overset{\circ}{P}$. P is called *open* if $P = \overset{\circ}{P}$.

(b) The point $x \in X$ is called a *closure point* of P if for all $\varepsilon > 0$ there is a point $p \in P$ satisfying $d(x,p) < \varepsilon$. The collection of all closure points of P is denoted by $\bar{P}$. P is called *closed* if $P = \bar{P}$.

LEMMA 1.16. *A subset P of a metric space is closed if and only if every convergent sequence with elements in P has its limit in P.*

THEOREM 1.17. *Let B be a Banach space and suppose $X \subset B$. Then X is a Banach space if and only if X is closed.*

DEFINITION 1.18. A series $\{\sum_{k=1}^n x_k\}$ in a normed space is called *absolutely convergent* provided $\sum_{n\in\mathbb{N}} \|x_n\| < \infty$.

THEOREM 1.19. *A normed space is complete if and only if every absolutely convergent series in it has a limit in it.*

1.2. L^p and Hardy Spaces

Let (X, Ω, μ) be a measure space. Define for $0 < p < \infty$

$$\|f\|_p = \left\{\int_X |f|^p \mathrm{d}\mu\right\}^{\frac{1}{p}}$$

and let $L^p = L^p(X,\mu) = \{f : \|f\|_p < \infty\}$.

LEMMA 1.20 (Arithmetic-Geometric Mean Inequality). *If $a, b \geq 0$ and $0 < \lambda < 1$, then $a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b$.*

THEOREM 1.21 (Hölder and Minkowski). *Let $p \geq 1$. Then*

(a) *If $f \in L^p$ and $g \in L^q$, where $\frac{1}{p} + \frac{1}{q} = 1$, then $fg \in L^1$ and $\int |fg| \mathrm{d}\mu \leq \|f\|_p \|g\|_q$;*

(b) *If $f, g \in L^p$, then $f + g \in L^p$ and $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.*

REMARK 1.22. Two numbers $p, q \geq 1$ which satisfy $\frac{1}{p} + \frac{1}{q} = 1$ are called *conjugate exponents*.

THEOREM 1.23. *Suppose $1 \le p < \infty$. Then $(L^p, \|\cdot\|_p)$ is a Banach space.*

DEFINITION 1.24. The *essential sup norm* is defined by

$$\|f\|_\infty = \inf\{a : \ \mu(\{x : \ |f(x)| > a\}) = 0\}.$$

The space $L^\infty = \{f : \ \|f\|_\infty < \infty\}$ is the set of all essentially bounded measurable functions.

EXAMPLE 1.25. (a) Let

$$f(t) = \begin{cases} 1 - t^2 & t \in [-1,1] \setminus \{0\} \\ 2 & t = 0. \end{cases}$$

Then $\|f\|_\infty = 1$.

(b) Let f be the characteristic function of the set $\mathbb{Q}$. Then $\|f\|_\infty = 0$.

REMARK 1.26. With the notation from Definition 1.24, Theorem 1.23 holds also for $p = \infty$.

To consider another example of a Banach space, let Δ denote the unit disc $\{z \in \mathbb{C} : \ |z| < 1\}$, and $\partial\Delta = \{z \in \mathbb{C} : \ |z| = 1\}$. Consider functions $f \in L^p(\partial\Delta)$, for example

$$f(e^{i\theta}) = e^{i\theta} \quad \text{or} \quad f(e^{i\theta}) = \sum_{k=-n}^{n} a_k e^{ik\theta}.$$

The nth Fourier coefficient of f is defined by

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} \mathrm{d}\theta.$$

[For example, if $f(e^{i\theta}) = e^{i\theta}$, then $\hat{f}(1) = 1$ and $\hat{f}(n) = 0$ for all other n.] The Fourier series associated with f is

$$f \sim \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{in\theta}.$$

DEFINITION 1.27. Let $p \ge 1$. The *Hardy Space H^p* is defined by

$$H^p = H^p(\partial\Delta) = \left\{ f \in L^p(\partial\Delta) : \ \hat{f}(-\mathbb{N}) = \{0\} \right\}.$$

REMARK 1.28. The Hardy space H^p consists of those functions in $L^p(\partial\Delta)$ whose negative Fourier coefficients all vanish. So the Fourier series for $f \in H^p$ looks as $f \sim \sum_{n=0}^{\infty} \hat{f}(n) e^{in\theta}$. We identify this series with $\sum_{n=0}^{\infty} \hat{f}(n) z^n$ where $z \in \partial\Delta$.

EXAMPLE 1.29. The function f defined by $f(e^{i\theta}) = e^{-i\theta}$ is not an element of H^p since $\hat{f}(-1) = 1$.

THEOREM 1.30. *Suppose* $1 \le p \le \infty$. *Then* $(H^p, \|\cdot\|_p)$ *is a Banach space.*

1.3. Linear Operators

DEFINITION 1.31. Let X, Y be vector spaces. A mapping $L : X \to Y$ is called a *linear operator* if it satisfies

$$L(\alpha x + y) = \alpha L(x) + L(y) \quad \text{for all} \quad x, y \in X,\ \alpha \in \mathbb{F}.$$

If $Y = \mathbb{F}$, then a linear operator is called a *linear functional.*

EXAMPLE 1.32. (a) Let $L : c \to \mathbb{R}$ be defined by $L(x) = \lim_{n\to\infty}$, where $x = (\xi_n) \in c$. Then L is linear.

(b) Let A be an $m \times n$-matrix with real entries. Then $L : \mathbb{R}^m \to \mathbb{R}^n$ defined by $L(x) = Ax$ for all $x \in \mathbb{R}^m$ is a linear operator.

DEFINITION 1.33. Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be normed linear spaces. Suppose that $L : \mathcal{X} \to \mathcal{Y}$ is a linear operator. Then

$$\|L\| = \sup_{x \in \mathcal{X}\setminus\{0\}} \frac{\|Lx\|_{\mathcal{Y}}}{\|x\|_{\mathcal{X}}}$$

is called the *norm* of the operator L. L is said to be *bounded* provided $\|L\| < \infty$. The collection of all bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$. The set $\mathcal{X}^* = \mathcal{B}(\mathcal{X}, \mathbb{F})$ is called the *dual space* of $\mathcal{X}$.

THEOREM 1.34. *With the notation from Def. 1.33,* $(\mathcal{B}(\mathcal{X}, \mathcal{Y}), \|\cdot\|)$ *is a normed linear space.*

EXAMPLE 1.35. (a) For $L : c \to \mathbb{R}$ defined by $L(x) = \lim_{n\to\infty} x_n$ for $x = (x_n) \in c$ we have $\|L\| = 1$.

(b) For $L : C[0, a] \to C[0, b]$ defined by $(Lx)(s) = s \int_0^a x(t)\mathrm{d}t$ for all $x \in C[0, a]$ we have $\|L\| = ab$.

(c) Let $\mathcal{X} = \mathcal{Y} = L^1(\mathbb{R})$, $g \in \mathcal{X}$, and define $L_g : \mathcal{X} \to \mathcal{Y}$ by $(L_g(f))(t) = \int_{-\infty}^{\infty} g(t-s)f(s)\mathrm{d}s$. Then L_g is a linear operator. L_g is bounded because of $\|L_g(f)\|_1 \le \|f\|_1\|g\|_1$.

(d) Let p and q be conjugate exponents. Fix $g \in L^q = L^q(X, \mu)$ and define $L_g : L^p \to \mathbb{C}$ by $L_g(f) = \int_X fg\mathrm{d}\mu$. Then L_g is a linear functional. L_g is bounded because of $|L_g(f)| \le \|f\|_p\|g\|_q$. In fact, $\|L_g\| = \|g\|_q$.

(e) Let P be the set of polynomials (as a subset of $(C[0,1], \|\cdot\|_\infty)$), and define $D : P \to P$ by $D(f) = f'$. Then D is a linear operator but not bounded.

THEOREM 1.36. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be normed linear spaces and* $L : \mathcal{X} \to \mathcal{Y}$ *be a linear operator. Then the following statements are equivalent:*

(a) *L is bounded;*
(b) *L is continuous;*
(c) *L is continuous at one point.*

THEOREM 1.37. *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed linear spaces. If $\mathcal{Y}$ is a Banach space, then so is $\mathcal{B}(\mathcal{X}, \mathcal{Y})$.*

COROLLARY 1.38. *Let $\mathcal{X}$ be a normed linear space. Then $\mathcal{X}^*$ is a Banach space.*

EXAMPLE 1.39 (The Dual of l^1). The space $(l^1)^*$ consists exactly of all functionals f defined by $f(x) = \sum_{n\in\mathbb{N}} c_n x_n$, where $x = \{x_n\} \in l^1$ and $c = \{c_n\} \in l^\infty$. Also, this relation implies $\|f\| = \|c\|_\infty$.

THEOREM 1.40. *Let φ be a linear functional on a normed space. Then φ is continuous if and only if $Ker\varphi$ is closed.*

THEOREM 1.41. (a) *All finite dimensional subspaces of a normed space are complete (and hence closed);*
(b) *All linear functionals on a finite dimensional normed space are continuous.*

CHAPTER 2

The Basic Principles

2.1. The Hahn–Banach Theorem

DEFINITION 2.1. Let X be a nonempty set. A *partial ordering* on X is a relationship "$\leq$" so that for all x, y, z each of the following holds:

(a) $x \leq x$;
(b) $x \leq y$ and $y \leq x$ imply $x = y$;
(c) $x \leq y$ and $y \leq z$ imply $x \leq z$.

An element $y \in X$ with $y \leq x$ for all $x \in E$ (where $E \subset X$) is called a *lower bound* of E. An element $y \in X$ that satisfies $x = y$ for all $x \in X$ with $x \leq y$ is called a *minimal element* of X. A subset E of X is said to be *linearly ordered* if for each two elements $x, y \in E$ we have either $x \leq y$ or $y \leq x$.

LEMMA 2.2 (Zorn's Lemma). *Let X be a partially ordered set such that every linearly ordered subset of X has an upper bound in X. Then X has a maximal element.*

DEFINITION 2.3. Let $\mathcal{X}$ be a real vector space. A *sublinear functional* is a mapping $p : \mathcal{X} \to \mathbb{R}$ such that the following holds:

(a) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in \mathcal{X}$;
(b) $p(\alpha x) = \alpha p(x)$ for all $x \in \mathcal{X}$ and all $\alpha \geq 0$.

THEOREM 2.4 (Hahn-Banach Theorem, Real Version). *Let $\mathcal{X}$ be a real vector space, p a sublinear functional on $\mathcal{X}$, $\mathcal{M}$ a subspace of $\mathcal{X}$, and f a linear functional on $\mathcal{M}$ such that $f(x) \leq p(x)$ for all $x \in \mathcal{M}$. Then there exists a linear functional F on $\mathcal{X}$ such that $F(x) \leq p(x)$ for all $x \in \mathcal{X}$ and $F(x) = f(x)$ for all $x \in \mathcal{M}$.*

DEFINITION 2.5. If $\mathcal{X}$ is a vector space over $\mathbb{F}$, a *seminorm* is a function $p : \mathcal{X} \to [0, \infty)$ having the properties:

(a) $p(x) \geq 0$ for all $x \in \mathcal{X}$;
(b) $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{F}$ and $x \in \mathcal{X}$;
(c) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in \mathcal{X}$.

THEOREM 2.6 (Hahn-Banach Theorem). *Let $\mathcal{X}$ be a vector space, p a seminorm on $\mathcal{X}$, $\mathcal{M}$ a subspace of $\mathcal{X}$, and f a linear functional on $\mathcal{M}$ such that $|f(x)| \leq p(x)$ for all $x \in \mathcal{M}$. Then there exists a linear functional F on $\mathcal{X}$ such that $|F(x)| \leq p(x)$ for all $x \in \mathcal{X}$ and $F(x) = f(x)$ for all $x \in \mathcal{M}$.*

THEOREM 2.7 (Hahn-Banach Theorem). *Let $\mathcal{X}$ be a normed linear space, $\mathcal{M}$ a subspace of $\mathcal{X}$, and f a bounded linear functional on $\mathcal{M}$. Then there exists a bounded linear functional F on $\mathcal{X}$ with $\|F\| = \|f\|$ and $F(x) = f(x)$ for all $x \in \mathcal{M}$.*

THEOREM 2.8 (Separation Theorem). *Let $\mathcal{X}$ be a normed linear space, $\mathcal{M}$ a subspace of $\mathcal{X}$, and $z \in \mathcal{X}$ with $\delta = \inf_{x \in \mathcal{M}} \|x - z\| > 0$. Then there exists a bounded linear functional F on $\mathcal{X}$ that satisfies $F(\mathcal{M}) = \{0\}$, $F(z) = \delta$, and $\|F\| = 1$.*

COROLLARY 2.9. *Let $\mathcal{X}$ be a normed linear space and $x_0 \in \mathcal{X} \setminus \{0\}$. Then there exists a bounded linear functional F on $\mathcal{X}$ with $F(x_0) = \|x_0\|$ and $\|F\| = 1$.*

COROLLARY 2.10. *Let $\mathcal{X}$ be a normed linear space. Then $\mathcal{X}^*$ separates points of $\mathcal{X}$ (i.e., for $x, y \in \mathcal{X}$, $x \neq y$, there exists $f \in \mathcal{X}^*$ such that $f(x) \neq f(y)$).*

2.2. The Uniform Boundedness Principle

DEFINITION 2.11. A subset E of a metric space is called *dense* if $\bar{E} = X$ and *nowhere dense* if $\overset{\circ}{\bar{E}} = \emptyset$. X is said to be of *first category* if it is the countable union of nowhere dense sets; otherwise, X is said to be of *second category.*

THEOREM 2.12 (Baire Category Theorem). *Let X be a complete metric space.*

(a) *If $\{U_n\}_{n \in \mathbb{N}}$ is a sequence of open dense subsets of X, then $\cap_{n \in \mathbb{N}} U_n$ is dense in X.*
(b) *X is not a countable union of nowhere dense sets.*

["A complete metric space is of second category."]

THEOREM 2.13 (Uniform Boundedness Principle). *Let $\mathcal{X}$ be a Banach space and let $\mathcal{Y}$ be a normed linear space. Let $\{L_\alpha\}_{\alpha \in I}$ be a collection of bounded linear operators $L_\alpha : \mathcal{X} \to \mathcal{Y}$. If $\sup_{\alpha \in I} \|L_\alpha(x)\| < \infty$ for all $x \in \mathcal{X}$, then $\sup_{\alpha \in I} \|L_\alpha\| < \infty$. ["Pointwise boundedness implies uniform boundedness."]*

THEOREM 2.14 (Banach-Steinhaus Theorem). *Let $\mathcal{X}$ be a Banach space and let $\mathcal{Y}$ be a normed linear space. Let $L_n : \mathcal{X} \to \mathcal{Y}$ be bounded linear operators for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} L_n(x)$ exists for each $x \in \mathcal{X}$, then $L(x) = \lim_{n\to\infty} L_n(x)$ defines a bounded linear operator from $\mathcal{X}$ to $\mathcal{Y}$.*

2.3. The Open Mapping and Closed Graph Theorems

DEFINITION 2.15. A linear map $L : \mathcal{X} \to \mathcal{Y}$ is called *open* if $L(U)$ is open in $\mathcal{Y}$ for each set U which is open in $\mathcal{X}$.

THEOREM 2.16 (Open Mapping Theorem). *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and suppose that $L : \mathcal{X} \to \mathcal{Y}$ is a bounded linear operator that is onto (i.e., $L(\mathcal{X}) = \mathcal{Y}$). Then L is an open mapping.*

THEOREM 2.17 (Banach Inverse Theorem). *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and suppose that $L : \mathcal{X} \to \mathcal{Y}$ is a bounded, bijective, linear operator. Then L has a bounded inverse.*

DEFINITION 2.18. The *graph* Γ of a linear map $L : X \to Y$ is defined to be $\Gamma(L) = \{(x, L(x)) : x \in X\} \subset X \times Y$. We say that L is *closed* if it has a closed graph.

EXAMPLE 2.19. If f is continuous, then $\Gamma(f)$ is closed.

THEOREM 2.20 (Closed Graph Theorem). *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and suppose that $L : \mathcal{X} \to \mathcal{Y}$ is a linear operator. If the graph of L is closed in $\mathcal{X} \times \mathcal{Y}$, then L is continuous.*

CHAPTER 3

Hilbert Spaces

3.1. Inner Product Spaces

DEFINITION 3.1. Let H be a vector space. An *inner product* on H is a mapping $\langle \cdot, \cdot \rangle : H^2 \to \mathbb{F}$ with

(a) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in H$;
(b) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in H$;
(c) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for all $x, y \in H$ and $\alpha \in \mathbb{F}$;
(d) $\langle x, x \rangle \geq 0$; $\langle x, x \rangle = 0$ iff $x = 0$.

H (together with $\langle \cdot, \cdot \rangle$) is then called an *inner product space.*

EXAMPLE 3.2. (a) $\langle f, g \rangle = \int_\Omega f \bar{g} \mathrm{d}\mu$ defines an inner product on $L^2(\Omega, \mu)$.
(b) $\langle x, y \rangle = \sum_{n \in \mathbb{N}} x_n \overline{y_n}$ defines an inner product on l^2.

THEOREM 3.3 (Cauchy-Schwarz Inequality). *Let H be an inner product space. Then $|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$ for all $x, y \in H$. Moreover, equality occurs iff $y = 0$ or $x = \lambda y$ for some constant λ.*

COROLLARY 3.4. *If H (together with $\langle \cdot, \cdot \rangle$) is an inner product space, then $\|x\| = \sqrt{\langle x, x \rangle}$ defines a norm on H.*

DEFINITION 3.5. A complete inner product space is called a *Hilbert space.*

THEOREM 3.6. *Let E be a nonempty, closed, and convex subset of a Hilbert space. Then E contains a unique element of least norm.*

REMARK 3.7. Let H be a Hilbert Space. For each fixed $z \in \mathcal{H}$, define $\varphi_z(x) = \langle x, z \rangle$ for all $x \in \mathcal{H}$. Then $\varphi_z \in \mathcal{H}^*$. We will see later that all $\varphi \in \mathcal{H}^*$ arise this way (Riesz Representation Theorem), so we will write $\mathcal{H}^* = \mathcal{H}$ ($\mathcal{H}$ is self-dual and reflexive). We also write $E^\perp = \{y \in \mathcal{H} : \langle x, y \rangle = 0 \text{ for all } x \in E\}$ for $E \subset \mathcal{H}$. If $\langle x, y \rangle = 0$, then we write $x \perp y$. If $x \in E^\perp$, we also write $x \perp E$.

THEOREM 3.8 (Projection Theorem). *Suppose $\mathcal{M}$ be a closed subspace of a Hilbert space $\mathcal{H}$. Let $h \in \mathcal{H}$. If f_0 is the unique element of $\mathcal{M}$ such that $d(h, \mathcal{M}) = d(h, f_0)$, then $h - f_0 \perp \mathcal{M}$. Conversely, if $f_0 \in \mathcal{M}$ with $h - f_0 \perp \mathcal{M}$, then $d(h, \mathcal{M}) = d(h, f_0)$.*

DEFINITION 3.9. Let $\mathcal{M}$ and $\mathcal{N}$ be two subspaces of $\mathcal{H}$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$ and such that each $x \in \mathcal{H}$ can be written as $x = x_M + x_N$ for $x_M \in \mathcal{M}$ and $x_N \in \mathcal{N}$. Then $\mathcal{H}$ is said to be the *orthogonal direct sum* of $\mathcal{M}$ and $\mathcal{N}$. We write $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$.

THEOREM 3.10. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ a nonempty closed subspace of $\mathcal{H}$. Then $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$.*

REMARK 3.11. According to the previous theorem, for each $h \in \mathcal{H}$ there are unique $Ph \in \mathcal{M}$ and $q \in \mathcal{M}^\perp$ with $h = Ph + q$. This defines an operator $P : \mathcal{H} \to \mathcal{H}$, which is called the *orthogonal projection* of h onto $\mathcal{M}$. It is easy to show that P is a bounded linear projection with $\|P\| = 1$.

THEOREM 3.12 (Riesz Representation Theorem). *Let $\mathcal{H}$ be a Hilbert space and let $f \in \mathcal{H}^*$. Then there exists a unique $z \in \mathcal{H}$ such that $f(x) = \langle x, z \rangle$ for all $x \in \mathcal{H}$. Moreover $\|z\| = \|f\|$.*

DEFINITION 3.13. An orthonormal subset of a Hilbert space $\mathcal{H}$ is a subset E having the properties

(a) $\|e\| = 1$ for all $e \in E$;
(b) $e_1 \perp e_2$ for all $e_1, e_2 \in E$ with $e_1 \neq e_2$.

EXAMPLE 3.14. (a) For $\mathcal{H} = \mathbb{F}^n$, the n kth unit vectors in $\mathbb{F}^n$ form an orthonormal subset of $\mathcal{H}$.
(b) For $\mathcal{H} = L^2[0, 2\pi]$, the functions e_m defined by $e_m(t) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{int}$ form an orthonormal subset of $\mathcal{H}$.

LEMMA 3.15. *Let $\{e_1, \ldots, e_n\}$ be an orthonormal set in a Hilbert space $\mathcal{H}$ and let $\mathcal{M} = \langle e_1, \ldots, e_n \rangle$. If P is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$, then $Ph = \sum_{k=1}^{n} \langle h, e_k \rangle e_k$ for all $h \in \mathcal{H}$.*

THEOREM 3.16 (Bessel's Inequality). *If $\{e_n : n \in \mathbb{N}\}$ is an orthonormal set and $h \in \mathcal{H}$, then $\sum_{n \in \mathbb{N}} | \langle h, e_n \rangle |^2 \leq \|h\|^2$.*

DEFINITION 3.17. An orthonormal set in a Hilbert space is called *complete* if the only vector which is orthogonal to every member of the set is the zero vector.

THEOREM 3.18. *Let $\{e_n : n \in \mathbb{N}\}$ be a complete orthonormal set in a Hilbert space $\mathcal{H}$. Then $x = \sum_{n \in \mathbb{N}} \langle x, e_n \rangle e_n$ and $\|x\|^2 = \sum_{n \in \mathbb{N}} | \langle x, e_n \rangle |^2$ for all $x \in \mathcal{H}$.*

3.2. Adjoint Operators

THEOREM 3.19. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Then there exists a unique $A^* \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that $\langle Ax, y \rangle = \langle x, A^* y \rangle$ for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$.*

DEFINITION 3.20. The operator A^* from Theorem 3.19 is called the *adjoint* of A. If $A^* = A$, then we call A *self-adjoint* (or *Hermitian*). If $A^*A = AA^*$, then we call A *normal*. If $A^*A = AA^* = I$, then we call A *normal*.

EXAMPLE 3.21. (a) If A is a square matrix with complex entries, then $A^* = \bar{A}^T$.
(b) Fix $\phi \in L^\infty(\mu)$. If $M_\phi : L^2(\mu) \to L^2(\mu)$ is defined by $M_\phi(f) = \phi f$, then $M_\phi^* = M_{\bar{\phi}}$.
(c) If $V : L^2([0,1]) \to L^2([0,1])$ is defined by $(Vf)(x) = \int_0^x f(t)\mathrm{d}t$, then V^* is given by $(V^*g)(t) = \int_t^1 g(t)\mathrm{d}t$.

REMARK 3.22. (a) $A^{**} = A$ and $\|A^*\| = \|A\|$.
(b) $(\alpha I)^* = \bar{\alpha} I$, where I is the identity operator.
(c) $(AB)^* = B^*A^*$.
(d) $(\operatorname{Im} A)^\perp = \operatorname{Ker} A^*$.

THEOREM 3.23. *Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H}, \mathcal{H})$. Then*
(a) $\|A\| = \sup_{\|y\| \le 1} \sup_{\|x\| \le 1} \operatorname{Re} \langle Ax, y \rangle$,
(b) *and, if A is self-adjoint,* $\|A\| = \sup_{\|x\|=1} |\langle Ax, x\rangle|$.

COROLLARY 3.24. *Let A be self-adjoint and suppose that $\langle Ah, h\rangle = 0$ holds for all h. Then $A = 0$.*

LEMMA 3.25. *An operator A on a Hilbert space is normal iff $\|Ah\| = \|A^*h\|$ holds for all h iff (provided $\mathbb{F} = \mathbb{C}$) the real and imaginary parts of A (i.e., $(A + A^*)/2$ and $(A - A^*)/(2\mathrm{i})$, respectively) commute.*

3.3. The Spectral Theorem

DEFINITION 3.26. Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. Then $\sigma(A) = \{\lambda \in \mathbb{C} : \ \lambda I - A \text{ is not invertible}\}$ is called the *spectrum* of A.

REMARK 3.27. There are two ways for λ to be in $\sigma(A)$: Either if $\lambda I - A$ is not onto, or if $\lambda I - A$ is not one-to-one. The last case is equivalent to the existence of an $x \in \mathcal{H} \setminus \{0\}$ with $(\lambda I - A)x = 0$, i.e., $Ax = \lambda x$. In this case we say that λ is an *eigenvalue* of A with corresponding *eigenvector* x.

THEOREM 3.28. *Suppose A is a self-adjoint bounded operator on a Hilbert space. Then all eigenvalues are real and eigenvectors corresponding to distinct eigenvalues are orthogonal.*

DEFINITION 3.29. Suppose X and Y are normed spaces and $A : X \to Y$ is linear. Then A is said to be *compact* if the sequence $\{Ax_n\}_{n\in\mathbb{N}} \subset Y$ has a convergent subsequence in Y whenever $\{x_n\}_{n\in\mathbb{N}} \subset X$ is a bounded sequence in X.

EXAMPLE 3.30. (a) Compact operators are bounded.
(b) Bounded finite rank perators are compact.
(c) The identity operator on an infinite-dimensional Hilbert space is not compact.

THEOREM 3.31. *Let K be a compact Hermitian operator on a Hilbert space. Then either $\|K\|$ or $-\|K\|$ is an eigenvalue of K.*

THEOREM 3.32. *Let K be a compact Hermitian operator on a Hilbert space. The set of eigenvalues of K is a set of real numbers which either is finite or consists of a countable sequence tending to zero.*

DEFINITION 3.33. A closed subspace $\mathcal{M}$ of a Hilbert space is said to be *invariant* under a bounded linear operator T if $T(\mathcal{M}) \subset \mathcal{M}$.

LEMMA 3.34. *Let $\mathcal{M}$ be a closed linear subspace of a Hilbert space which is invariant under a bounded linear operator T. Then $\mathcal{M}^\perp$ is invariant under T^*.*

THEOREM 3.35 (Spectral Theorem). *Let K be a compact Hermitian operator on a Hilbert spcae $\mathcal{H}$. Then there exists a finite or infinite orthonormal sequence $\{\varphi_n\}$ of eigenvectors of K, with corresponding eigenvectors $\{\lambda_n\}$, such that $Kx = \sum_n \lambda_n \langle x, \varphi_n \rangle \varphi_n$ for all $x \in \mathcal{H}$. The sequence $\{\lambda_n\}$, if infinite, tends to 0.*