1. (a) Let $(\mathbb{R}, d)$ be a metric space. Show that $(\mathbb{R}, D)$ with $D(x,y) = \frac{d(x,y)}{1+d(x,y)}$ is also a metric space.
   (b) Show that the set of binary words with length n together with the so-called Hamming distance (which is the number of places where the two words differ from each other) is a metric space.
2. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. Show that the following mappings are continuous:
   (a) $\|\cdot\| : \mathcal{X} \to \mathbb{R}$;
   (b) $+ : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$;
   (c) $\cdot : \mathbb{F} \times \mathcal{X} \to \mathcal{X}$.
3. A subset E of a vector space is *convex* if for every $x, y \in E$, $\lambda x + (1-\lambda)y \in E$ for all $0 \le \lambda \le 1$. Let $\mathcal{X}$ be a normed space. Prove the following:
   (a) The open unit ball $\{x \in \mathcal{X} : \|x\| < 1\}$ is convex;
   (b) For a convex set $P \subset \mathcal{X}$, $\bar{P}$ is convex;
   (c) For a convex set $P \subset \mathcal{X}$, $\overset{\circ}{P}$ is convex.
4. Let $\mathcal{X}$ be a Banach space, and suppose that there exists $q < 1$ such that the mapping $T : \mathcal{X} \to \mathcal{X}$ satisfies $\|T(x) - T(y)\| \le q\,\|x - y\|$ for all $x, y \in \mathcal{X}$.
   (a) Show that T is continuous.
   (b) Let $x_0 \in \mathcal{X}$ and define a sequence $\{x_n\}$ recursively by $x_{n+1} = T(x_n)$ for all $n \in \mathbb{N}_0$. Show that $\tilde{x} = \lim_{n\to\infty} x_n$ exists and satisfies $T(\tilde{x}) = \tilde{x}$.
5. Let a linear system of equations $\xi - A\xi = \eta$ be given. Suppose that A is an $n \times n$-matrix such that the sum of the squares of all its entries is less than 1.
   (a) Apply the previous problem to suggest an iterative method to solve this system.
   (b) Test your method with the system $\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{3} & \frac{3}{4} \end{pmatrix} \xi = \begin{pmatrix} 0 \\ \frac{7}{6} \end{pmatrix}$. If you like (for 10 extra points) write a computer program that implements your method.
6. (a) Show that l_0 is not a closed linear subspace of $(l_2, \|\cdot\|_2)$.
   (b) Let $x : [a,b] \to \mathbb{R}$. Show $\|x\|_\infty \le \|x\|_{BV}$.
7. (a) Let l^∞ be the set of all bounded real sequences. Show that $(l^\infty, \|\cdot\|_\infty)$ is a Banach space.
   (b) Show that the set c of all convergent real sequences (with norm $\|\cdot\|_\infty$) is a Banach space.
8. Let $p \in (0,1)$. Show that
   (a) $(L^p[0,1], \|\cdot\|_p)$ is not a normed space;
   (b) $d(f,g) = \int |f-g|^p \mathrm{d}\mu$ is a metric on L^p.
9. For $z \in \partial\Delta = \{z \in \mathbb{C} : |z| = 1\}$, define $f(z) = \frac{1}{\sqrt{1-z}}$.
   (a) Is $f \in L^1(\partial\Delta)$?
   (b) Is $f \in L^2(\partial\Delta)$?