37. Let $\mathcal{X}$ be a linear space. Elements of $\mathcal{X}/\mathcal{M}$ (with $\mathcal{M}$ a subspace) are also called *linear varieties*. A *hyperplane* $\mathcal{H}$ is a a maximal proper linear variety. Show the following:
    (a) Let $\mathcal{H}$ be a hyperplane in $\mathcal{X}$. Then there exists a linear functional $f$ on $\mathcal{X}$ and a constant $c$ such that $\mathcal{H} = \{x \in \mathcal{X} : f(x) = c\}$. Conversely, if $f$ is a nonzero linear functional on $\mathcal{X}$, the set $\{x \in \mathcal{X} : f(x) = c\}$ is a hyperplane in $\mathcal{X}$.
    (b) Let $\mathcal{H}$ be a hyperplane in $\mathcal{X}$. If $\mathcal{H}$ does not contain the origin, there is a unique linear functional $f$ on $\mathcal{X}$ such that $\mathcal{H} = \{x \in \mathcal{X} : f(x) = 1\}$.
    (c) Let $f$ be a nonzero linear functional on $\mathcal{X}$. Then the hyperplane $\{x \in \mathcal{X} : f(x) = c\}$ is closed for every $c$ if and only if $f$ is continuous.
    (d) If $f$ is a nonzero continuous linear function on $\mathcal{X}$, then the so-called negative and positive *half-spaces* $\{x \in \mathcal{X} : f(x) \le c\}$, $\{x \in \mathcal{X} : f(x) < c\}$, $\{x \in \mathcal{X} : f(x) \ge c\}$, $\{x \in \mathcal{X} : f(x) > c\}$ are closed, open, closed, and open, respectively.
38. Prove the following "geometric" form of the Hahn-Banach Theorem and its consequences:
    (a) Let $K$ be a convex set having a nonempty interior in a real normed linear space $\mathcal{X}$. Suppose $V$ is a linear variety in $\mathcal{X}$ containing no interior points of $K$. Then there is a closed hyperplane in $\mathcal{X}$ containing $V$ but containing no interior points of $K$.
    (b) If $x$ is not an interior point of a convex set $K$ which contains interior points, there is a closed hyperplane $\mathcal{X}$ containing $x$ such that $K$ lies on one side of $\mathcal{H}$.
    (c) Let $K_1$ and $K_2$ be convex sets in $\mathcal{X}$ such that $K_1$ has interior points and $K_2$ contains no interior point of $K_1$. Then there is a closed hyperplane $\mathcal{X}$ separating $K_1$ and $K_2$ (i.e., $K_1$ and $K_2$ lie in opposite half-spaces).
    (d) If $K$ is a closed convex set and $x \notin K$, there is a closed half space containing $K$ but not $x$.
    (e) If $K$ is a closed convex set in a normed space, then $K$ is equal to the intersection of all the closed half-spaces that contain it.
39. Let $X$ be a complete metric space and $E \subset X$.
    (a) If $E$ is of first category, is $E^c$ then of second category?
    (b) If $E$ is of second category, is $E^c$ then of first category?
40. Consider the space $l_0$ together with $\|\cdot\|_2$.
    (a) Show that $l_0$ is of first category. Conclude that $l_0$ is not complete.
    (b) Show that $T : l_0 \to l_0$ defined by $T(\{x_n\}) = \{\frac{x_n}{n}\}$ is a bijective, linear and bounded operator and that $T^{-1}$ is a linear but unbounded operator.
41. Show that any infinite dimensional Banach space has a subspace that is not closed.
42. Let $\mathcal{X}$ be a Banach space. Show that the following statements are equivalent:
    (a) $\mathcal{X}$ has finite dimension;
    (b) All subspaces of $\mathcal{X}$ are closed;
    (c) All linear functionals on $\mathcal{X}$ are continuous.
43. Let $\mathcal{X}$ be a Banach space and $\mathcal{Y}$ a normed linear space. Let $L_n \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ for each $n \in \mathbb{N}$ and suppose that $Lx = \lim_{n\to\infty} L_n x$ exists for each $x \in \mathcal{X}$. Prove the following "Fatou's Lemma": $L$ is a bounded linear operator and $\|L\| \le \liminf_{n\to\infty} \|L_n\|$.
44. Let $\mathcal{X}$ be a Banach space.
    (a) Suppose $P : \mathcal{X} \to \mathcal{X}$ is linear such that $P^2 = P$ and such that $\operatorname{Im} P$ and $\operatorname{Ker} P$ are closed. Show that $P$ is continuous.
    (b) A linear operator $P : \mathcal{X} \to \mathcal{X}$ satisfying $P^2 = P$ is called a *linear projection*. Show that $\mathcal{X}$ is finite dimensional if and only if all linear projections are continuous.