54. Let $A$ be a bounded linear operator acting between two real Hilbert spaces. Show the following relations:
    (a) $[\mathrm{Im}(A)]^\perp = \mathrm{Ker}(A^*)$ and $\overline{\mathrm{Im}(A)} = [\mathrm{Ker}(A^*)]^\perp$;
    (b) $[\mathrm{Im}(A^*)]^\perp = \mathrm{Ker}(A)$ and $\overline{\mathrm{Im}(A^*)} = [\mathrm{Ker}(A)]^\perp$.
55. Let $A$ be a bounded linear operator acting between two real Hilbert spaces. Assume that $\mathrm{Im}(A)$ is closed and show that among all vectors $x_1$ satisfying $\|Ax_1 - y\| = \min_x \|Ax - y\|$ there is a unique vector $x_0$ of minimum norm. The Moore-Penrose Inverse $A^\dagger$ of $A$ is the operator mapping $y$ into its corrsponding $x_0$. Show the following:
    (a) $A^\dagger$ is linear and bounded;
    (b) $A^\dagger A$ and $AA^\dagger$ are both self-adjoint;
    (c) $AA^\dagger A = A$ and $A^\dagger AA^\dagger = A^\dagger$.
56. Let $A$ be a bounded linear operator acting between two real Hilbert spaces. Assume that $\mathrm{Im}(A)$ is closed and prove that $A^\dagger = \lim_{\varepsilon\to 0+}(A^*A+\varepsilon I)^{-1}A^* = \lim_{\varepsilon\to 0+} A^*(AA^*+\varepsilon I)^{-1}$ holds. Use this formula to calculate the Moore-Penrose inverses of $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$.
57. (a) Let $r$ and $p$ be given real valued functions on $I = [0,1]$. Let $\mathcal{A}$ be the set of all twice differentiable real functions $y$ on $I$ such that $y(0) = y(1) = 0$. Find an operator $L$ such that $\int_0^1 (py^2 + r\dot{y}^2)(t)\mathrm{d}t = \langle Ly, y\rangle$ for all $y \in \mathcal{A}$.
    (b) Let $r$ and $p$ be given sequences on $I = \{0, 1, \dots, N\}$. Let $\mathcal{A}$ be the set of all real sequences on $I$ such that $y_0 = y_N = 0$. Find an operator $L$ such that $\sum_{n=0}^{N-1}\{p_n y_{n+1}^2 + r_n(\Delta y_n)^2\} = \langle Ly, y\rangle$ for all $y \in \mathcal{A}$.
    (c) Let $A$, $B$, and $C$ be real square matrix valued functions on $I = [0,1]$ such that $B$ and $C$ are self-adjoint. Let $\mathcal{A}$ be the set of all differentiable real vector valued functions $x$ on the interval $I = [0,1]$ such that $x(0) = x(1) = 0$ and such that there exists a $u$ with $\dot{x} = Ax + Bu$. Find an operator $L$ such that $\int_0^1 (x^TCx + u^TBu)(t)\mathrm{d}t = \langle Lx, x\rangle$ for all $x \in \mathcal{A}$.
    (d) Create a discrete analogue problem to (c) and solve it.
58. Let $\alpha$ be a complex sequence and define $A$ by $A(x_1, x_2, \dots) = (\alpha_1 x_1, \alpha_2 x_2, \dots)$. Characterize the sequences $\alpha$ that guarantee $A : l^2 \to l^2$, $A$ self-adjoint, normal, compact, and invertible. Find the spectrum of $A$.
59. Read Section II.6 of the textbook and work on all exercises there.
60. Read the notes on “Approximation by analytic functions” and work on as many exercises you like (at least one).
61. Read the “Afterword” to the book “An Introduction to Hilbert Space” by N. Young.