19. Prove that $\sqrt{3} \in \mathbb{R} \setminus \mathbb{Q}$.

20. Prove the following statements using the PMI:

    (a) $\forall n \in \mathbb{N}: \ n! \geq 2^{n-1}$;

    (b) $\forall n \in \mathbb{N} \setminus \{1,2\}: \ 2n+1 \leq 2^n$;

    (c) $\forall n \in \mathbb{N} \setminus \{1,2,3\}: \ 2^n \geq n^2$;

    (d) $\forall n \in \mathbb{N}: \ 3^n \geq n2^n$;

    (e) $\forall n \in \mathbb{N}: \ 11^{n+1} + 12^{2n-1}$ is divisible by 133;

    (f) $\forall n \in \mathbb{N} \ \forall x_1, \ldots, x_n \geq 0: \ \prod_{k=1}^{n} (1+x_k) \geq 1 + \sum_{k=1}^{n} x_k$.

21. Let $a, b \in \mathbb{R}$ with $a, b \geq 0$ and $n \in \mathbb{N}$. Show the following:

    (a) $a \leq b \iff a^n \leq b^n$;

    (b) $a^n - b^n \geq nb^{n-1}(a-b)$ provided $a \geq b$.

22. Let $n \in \mathbb{N}$. Find the values of the following sums:

    (a) $\sum_{k=1}^{n} \left(1 + \frac{1}{n}\right)^k$;

    (b) $\sum_{k=4}^{n} \left\{2\left(\frac{4}{5}\right)^k - \frac{2}{3^k}\right\}$;

    (c) $\sum_{k=1}^{n} \frac{1}{k(k+1)}$;

    (d) $\sum_{k=2}^{n} \frac{1}{k(k+2)}$;

    (e) $\sum_{\mu=1}^{n} \sum_{\nu=\mu+1}^{n} \frac{\mu^2}{\nu(2\nu-1)}$.

23. Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Find the values of the following sums:

    (a) $\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$;

    (b) $\sum_{k=0}^{n} \binom{n}{k} \frac{k}{n} x^k (1-x)^{n-k}$;

    (c) $\sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n}\right)^2 x^k (1-x)^{n-k}$.

24. Show that $\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$ holds for all $a, b \in \mathbb{R}$.

25. Find (if existent) min, max, sup, and inf of the following sets:

    (a) $M = \left\{\frac{1}{m} + \frac{1}{n}: \ m, n \in \mathbb{N}\right\}$;

    (b) $T = \left\{\frac{m-n}{m+n}: \ m, n \in \mathbb{N}\right\}$.

26. Let $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$ with $0 \leq k \leq n$. Show $\frac{n^k}{k!} \geq \binom{n}{k} \geq \frac{n^k}{k!}\left(1 - \frac{k(k-1)}{n}\right)$.