27. Give a direct ε/N-verification of the convergence of the following sequences:

(a) $a_n = \frac{2}{\sqrt{n}}$;

(b) $a_n = \frac{1}{n+3}$;

(c) $a_n = \frac{3}{\sqrt{n}} + \frac{2}{n} + 4$;

(d) $a_n = \frac{n^2}{n^2+n}$.

28. Let $\{a_n\}$ be a real sequence. We say $\lim\limits_{n\to\infty} a_n = \infty$ provided

$$\forall K > 0 \, \exists N \in \mathbb{N} \, \forall n \geq N : \; a_n > K.$$

Prove that $\lim\limits_{n\to\infty} \{n^3 - 4n^2 - 99n\} = \infty$.

29. Suppose $x_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} x_n = x_0 \in \mathbb{R}$. Show $\lim\limits_{n\to\infty} \sqrt{x_n} = \sqrt{x_0}$.

30. Prove the following statements:

(a) $\lim_{k\to\infty} a_k < c \quad \Longrightarrow \quad \exists\, N \in \mathbb{N} \, \forall\, k \geq N : \; a_k < c$;

(b) $\lim_{k\to\infty} a_k > c \quad \Longrightarrow \quad \exists\, N \in \mathbb{N} \, \forall\, k \geq N : \; a_k > c$.

31. Suppose $\{a_n\}$ is a real sequence and define $s_n = \frac{1}{n} \sum\limits_{k=1}^{n} a_k$ for $n \in \mathbb{N}$. Prove:

(a) If $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} s_n = 0$;

(b) If $\lim_{n\to\infty} a_n = a \in \mathbb{R}$, then $\lim_{n\to\infty} s_n = a$.

32. Discuss the convergence of each of the following sequences:

(a) $\frac{n^3-6n^2+1}{2n^3+5}$, $\frac{n^2-6n+1}{n^3+5}$, $\frac{n^3-6n^2+1}{n^2+5}$, $\frac{6^n-3^n}{2\cdot 6^n+3^n}$;

(b) $\sqrt{n+1} - \sqrt{n}$, $\left(\sqrt{n+1} - \sqrt{n}\right)\sqrt{n}$, $\left(\sqrt{n+1} - \sqrt{n}\right) n$;

(c) $\prod\limits_{k=2}^{n} \left(1 - \frac{1}{k^2}\right)$, $\sum\limits_{k=1}^{n} \frac{k^2}{n^3}$, $\sum\limits_{k=1}^{n} \frac{1}{kn}$.