39. Prove the following statements:

(a) If $a_n \to 0$ as $n \to \infty$, then $\left(1 + \frac{a_n}{n}\right)^n \to 1$ as $n \to \infty$.

(b) If $x \in \mathbb{R}$, then $\lim\limits_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$ exists (put $e(x) := \lim\limits_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$ for $x \in \mathbb{R}$).

(c) If $x \in \mathbb{R}$, then $e(x) > 0$ and $e(-x) = \frac{1}{e(x)}$.

(d) If $|x| < 1$, then $1 + x \le e(x) \le \frac{1}{1-x}$.

(e) If $a_n \to 0$ as $n \to \infty$, then $e(a_n) \to 1$ as $n \to \infty$.

(f) If $x, y \in \mathbb{R}$, then $e(x + y) = e(x)e(y)$.

(g) If $x, y \in \mathbb{R}$, then $e(x) > e(y) \iff x > y$.

(h) The function $e : \mathbb{R} \to (0, \infty)$ is continuous.

(i) The function $e : \mathbb{R} \to (0, \infty)$ is invertible (so $l := e^{-1} : (0, \infty) \to \mathbb{R}$ exists).

(j) If $x, y > 0$, then $l(xy) = l(x) + l(y)$.

(k) If $x, y > 0$, then $l(x) > l(y) \iff x > y$.

(l) For $a > 0$ and $x \in \mathbb{R}$, put $A(a, x) := e(xl(a))$. Let $a, b > 0$. Then $A(a, n) = a^n$ for all $n \in \mathbb{Z}$, $A\left(a, \frac{1}{2}\right) = \sqrt{a}$, and $A(a, x)A(a, y) = A(a, x + y)$, $A(A(a, x), y) = A(a, xy)$, $A(a, x)A(b, x) = A(ab, x)$ for all $x, y \in \mathbb{R}$.

(m) If $x \in \mathbb{R}$, then $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \frac{x^k}{k!}$ exists and equals $e(x)$.

(n) $\lim\limits_{n\to\infty} \left\{ \sum\limits_{k=1}^{n} \frac{1}{k} - l(n) \right\}$ exists. (For 5 points extra credit, use a computer to determine that limit up to six decimal places.)

(o) $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \frac{(-1)^k}{k+1} = l(2)$.