Remark 11.1
We let $T>0$ and write $S_n = S(nT)$ for integers $n$. We assume $S_{n+1}$ is either $S_n u$ with probability $p$ or $S_n d$ with probability $1-p$, where $0 < d < 1 < u$ and $0 < p < 1$ ("stock price follows random walk").

Example 11.2
Suppose $S_0 = 20$, $u = 1.1$, $d = 0.9$. Consider a European call on the stock with strike price $K = 21$ and maturity $T = 1/4$, $r = 0.12$. We seek a risk-less portfolio in stock and option.

Definition 11.3
By a European derivative security or contingent claim with stock $S$ as the underlying asset we mean a random variable of the form $f(S(T))$, where $f$ is a given function, called the payoff.

Example 11.4
• call: $f(S) = (S - K)^+$
• put: $f(S) = (K - S)^+$
• long forward: $f(S) = S - K$

Theorem 11.5
In the one-step binomial tree model, $f = E(f(S_1))e^{-rT}$ ("the present value of the contingent claim is equal to the discounted payoff expectation in a risk-neutral world, independent of $p$"). Here $p_* = (e^{rT} - d)/(u - d)$. 
Example 11.6
Suppose $S_0 = 20$, $u = 1.1$, $d = 0.9$, $f_u = 1$, $f_d = 0$, $r = 0.12$, $T = 0.25$. In the one-step BM, find $c$.

Theorem 11.7
In the $n$-step binomial tree model,
\[ f = E^*(f(S_N))e^{-rN} \]

Example 11.8
- Example 11.6 with $n=2$
- 2-year European put, $K=52$, $S_0=50$, $r=0.05$, $T=1$, $u=1.2$, $d=0.8$
- American put (as above)

Theorem 11.9 (Cox-Ross-Rubinstein formula)
In the binomial model, the price of a European call and put with strike price $K$ to be exercised after $N$ time steps is
\[
\begin{align*}
c &= S_0[1 - \Phi(m-1,N,q)] \\
&\quad - Ke^{-rNT}[1 - \Phi(m-1,N,p*)]
\end{align*}
\]
and
\[
\begin{align*}
p &= S_0\Phi(m-1,N,q) \\
&\quad + Ke^{-rNT}\Phi(m-1,N,p*)
\end{align*}
\]
where $\Phi$ is the cdf of the binomial distribution, $q = pue^{-rT}$, and $m$ is the smallest nonnegative integer with $S_0umdN-m > K$.

Remark 11.10
The expected return $\mu$ and the volatility $\sigma$ of a stock price will be defined in such a way that
\[
\begin{align*}
E(S_T) &= S_0e^{\mu T} \\
\text{Var}(S_T) &= S_0^2\sigma^2T
\end{align*}
\]
for small $T$ in a one-step BM. Then $u = e^{\sigma T/2}$ and $d = e^{-\sigma T/2}$.
Example 11.11

• Example 11.8(c), where we assume that volatility is 30% \( p = \frac{(a-d)}{(u-d)} \).

Example 11.11 (continued)

• For stocks paying a continuous dividend yield, use formulas as above except
  \[ a = e^{(r-q)T} \].
  E.g., stock index with \( S_0 = 810 \), \( \sigma = 0.2 \), \( q = 0.02 \), \( r = 0.05 \), \( K = 800 \), \( T = 0.25 \), \( N = 2 \), European call.

Example 11.11 (continued)

• Foreign currencies can be regarded as an asset providing a yield \( r_f \), use
  \[ a = e^{(r-r_f)T} \].
  E.g., AUD is currently worth 0.61 USD and this exchange rate has a volatility of 12%, \( r_f = 0.07 \), \( r = 0.05 \). Value a 3-month American call with strike price of 0.6 using a three-step tree.

Example 11.11 (continued)

• Options on futures: Futures price should have an expected growth rate of zero. Use previous formulas with \( a = 1 \).