Chapter 9

Properties of Stock Options

Proposition 9.1
If we let $c$, $p$, $C$, $P$ be the prices of a European call, European put, American call, American put, respectively (all on the same stock, same $K$ and same $T$), then $0 \leq c \leq C$ and $0 \leq p \leq P$.

Theorem 9.2
For a stock paying no dividends, $$(S_0 - Ke^{-rT})^+ \leq c < S_0.$$}

Example 9.3
For a European call on a non-dividend paying stock with $S_0=51$, $K=50$, $T=1/2$, $r=0.12$, find the bounds on $c$.

Theorem 9.4
For a stock paying no dividends, $$(Ke^{-rT} - S_0)^+ \leq p < Ke^{-rT}.$$}

Example 9.5
For a European put on a non-dividend paying stock with $S_0=38$, $K=40$, $T=1/4$, $r=0.1$, find the bounds on $p$. 


Remark 9.6

- Theorems 9.2 and 9.4 can be graphically summarized.
- Note also that by Corollary 5.12 (first part), $S_0 - Ke^{-rT} = V(0)$ is the value of a long forward contract on the non-dividend paying stock.

Theorem 9.7

Put-call parity holds:
$$c - p = S_0 - Ke^{-rT}.$$ 

Example 9.8

Suppose a stock paying no dividends trades at $15.60 a share. European calls on the stock with strike price $15 and exercise date in 3 months are trading at $2.83, $r=6.72\% (cc)$. What is the price of a European put on the same stock with the same strike price and the same exercise date?

Theorem 9.9

For a stock paying no dividends, $C = c$.

Corollary 9.10

For a stock paying no dividends, $(S_0 - Ke^{-rT})^+ C < S_0$.

Example 9.11

Suppose $S=10$, strike price of American put expiring in one year is $80$, and $r=0.16$. Here it is better to do early exercise.
Theorem 9.12
For a stock paying no dividends,
\[(K-S_0)^+ \leq P < K.\]

Theorem 9.13
Put-call parity estimates hold:
\[S_0 - K \leq C - P \leq S_0 - Ke^{-rT}\]
(for a stock paying no dividends).

Example 9.14
Suppose \(S_0 = 19, \quad K = 20, \quad T = 5/12, \quad r = 0.1, \quad C = 1.50.\) Find the bounds on \(P.\)

Theorem 9.15
For a stock paying dividends with present value \(D,\)
\[\begin{align*}
    (S_0 - D - Ke^{-rT})^+ & \leq c < S_0 - D \\
    (Ke^{-rT} + D - S_0)^+ & \leq p < Ke^{-rT} \\
    c - p = S_0 - D - Ke^{-rT} & \leq c - p < S_0 - D - Ke^{-rT}
\end{align*}\]

Theorem 9.15 (continued)
\[\begin{align*}
    S_0 - D - K & \leq c - p \leq S_0 - Ke^{-rT} \\
    \max(0, S_0 - D - Ke^{-rT}, S_0 - K) & \leq C < S_0 \\
    \max(0, Ke^{-rT} + D - S_0, K - S_0) & \leq P < K
\end{align*}\]

Remark 9.16
In the rest of this chapter, we discuss the dependence of \(c, p, C,\) \(P\) on \(K, S, T,\) and \(r\) (varying only one variable while keeping the others constant; all on stocks paying no dividends).
Proposition 9.17

c is nonincreasing in K while p is nondecreasing in K, i.e.,
\[ c(K) \geq c(K') \text{ and } p(K) \leq p(K') \]
for \[ K \leq K'. \]

Theorem 9.18

c and p are Lipschitz continuous in K, namely,
\[ |c(K)-c(K')| \leq e^{-rT}|K-K'| \]
and
\[ |p(K)-p(K')| \leq e^{-rT}|K-K'|. \]

Theorem 9.19

c and p are convex in K, i.e.,
\[ K \leq K' \text{ and } 0 < \alpha < 1 \]
implies
\[ c(\alpha K + (1-\alpha)K') \leq \alpha c(K) + (1-\alpha)c(K') \]
and
\[ p(\alpha K + (1-\alpha)K') \leq \alpha p(K) + (1-\alpha)p(K'). \]

Theorem 9.20

If \[ S \leq S' \], then
- \[ c(S) \leq c(S') \text{ and } p(S) \geq p(S') \]
- \[ c(S') - c(S) \leq S' - S \]
- \[ p(S) - p(S') \leq S' - S \]
- c and p are convex in S.

Theorem 9.21

For American options, Proposition 9.17 holds, Theorem 9.18 (with \( T=0 \) there) holds, and Theorems 9.19 and 9.20 hold. Finally, if \( T \leq T' \), then
\[ C(T) \leq C(T') \text{ and } P(T) \leq P(T'). \]

Remark 9.22

Summary of the effect on the stock option price when the parameter is increased while the other ones are kept constant:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>EC</th>
<th>EP</th>
<th>AC</th>
<th>AP</th>
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Summary Table:

- EC: Effect of Increasing
- EP: Effect of Positive
- AC: Effect of Adding
- AP: Effect of Positive

The table indicates how each parameter affects the stock option price. For example, increasing K (K) increases c and p, while increasing S (S) decreases both c and p.