Chapter 11

Binomial Trees

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Remark 11.1

We let $T>0$ and write $S_n=S(nT)$ for integers n. We assume S_{n+1} is either $S_n u$ with probability p or $S_n d$ with probability $1-p$, where

$0<d<1<u$ and $0<p<1$

("stock price follows random walk").

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Example 11.2

Suppose $S_0=20$, $u=1.1$, $d=0.9$. Consider a European call on the stock with strike price K=21 and maturity T=1/4, r=0.12. We seek a risk-less portfolio in stock and option.

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Definition 11.3

By a European derivative security or contingent claim with stock S as the underlying asset we mean a random variable of the form f(S(T)), where f is a given function, called the payoff.

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Example 11.4

- call: $f(S)=(S-K)^+$
- put: $f(S)=(K-S)^+$
- long forward: $f(S)=S-K$

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Theorem 11.5

In the one-step binomial tree model,

$f=E_*(f(S_1))e^{-rT}$

("the present value of the contingent claim is equal to the discounted payoff expectation in a risk-neutral world, independent of p"). Here $p_*=(e^{rT}-d)/(u-d)$.

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Example 11.6

Suppose $S_0=20$, $u=1.1$, $d=0.9$, $f_u=1$, $f_d=0$, $r=0.12$, $T=0.25$.

- In the one-step BM, find c.
- Find also p.
- Show that PCP holds.

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Theorem 11.7

In the n-step binomial tree model,

$f=E_*(f(S_n))e^{-rnT}$.

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Example 11.8

- Example 11.6 with n=2
- 2-year European put, $K=52$, $S_0=50$, $r=0.05$, $T=1$, $u=1.2$, $d=0.8$
- American put (as above)

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Theorem 11.9

(Cox-Ross-Rubinstein formula)

In the binomial model, the price of a European call and put with strike price K to be exercised after N time steps is

$$c=S_0[1-\Phi(m-1,N,q)] -Ke^{-rNT}[1-\Phi(m-1,N,p_*)]$$

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Theorem 11.9 (continued)

and

$$p=-S_0\Phi(m-1,N,q) +Ke^{-rNT}\Phi(m-1,N,p_*),$$

where Φ is the cdf of the binomial distribution, $q=p_*ue^{-rT}$, and m is the smallest nonnegative integer with

$$S_0u^md^{N-m}>K.$$

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Remark 11.10

The expected return μ and the volatility σ of a stock price will be defined in such a way that

$$E(S_1)=S_0e^{\mu T} \text{ and } Var(S_1)=S_0^2\sigma^2T$$

for small T in a one-step BM. Then

$$u=e^{\sigma T^{1/2}} \text{ and } d=e^{-\sigma T^{1/2}}$$

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Example 11.11

- Example 11.8(c), where we assume that volatility is 30%

  $p=(a-d)/(u-d)$.

Example 11.11 (continued)

- For stocks paying a continuous dividend yield, use formulas as above except

  $a=e^{(r-q)T}$.

  E.g., stock index with $S_0=810$, $\sigma=0.2$, $q=0.02$, $r=0.05$, $K=800$, $T=0.25$, $N=2$, European call.

Example 11.11 (continued)

- Foreign currencies can be regarded as an asset providing a yield r_f, use

  $a=e^{(r-r_f)T}$.

  E.g., AUD is currently worth 0.61 USD and this exchange rate has a volatility of 12%, $r_f=0.07$, $r=0.05$. Value a 3-month American call with strike price of 0.6 using a three-step tree.

Example 11.11 (continued)

- Options on futures: Futures price should have an expected growth rate of zero. Use previous formulas with

  $a=1$.