Chapter 4

Interest Rates

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Remark 4.1

Types of rates are:

- Treasury rates (government, virtually risk free)
- LIBOR rates (1/3/6/12-month in all major currencies, not totally risk free)
- Repo-rates (very little credit risk)

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Definition 4.2

Let V(t) be the wealth at time t (years).

We talk about discrete or periodic compounding with frequency m times a year and interest rate r per annum provided

$V(t)=V(0)(1+r/m)^{mt}$ for all t≥0.

$(1+r/m)^{mt}$ is called the growth factor,

$(1+r/m)^{-mt}$ is called the discount factor.

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Example 4.3

- Let r=0.1. Find the value of $100 after 1 year with periodic compounding and m=1, 2, 4, 12, 52, 365.
- How long does it take to double a capital attracting interest at 6% daily?
- What is r if a deposit subject to annual compounding is doubled after 10 years?

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Example 4.3 (continued)

- How long will it take to earn $1 if r=0.1 (a.c.) and V(0)=1 cent?
- Pay $1000 every year after 1, 2, and 3 years. What is the present value of this payment stream? Use annual compounding at 25%.

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Definition 4.4

An annuity is a sequence of finitely many payments of a fixed amount due at equal time intervals.

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Proposition 4.5

For discrete annual compounding with rate r and payments of C every year, the present value of an annuity for n years is

$C(1-(1+r)^{-n})/r$.

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Example 4.6

Consider a loan of $1000 to be paid back in 5 equal installments due at yearly intervals. The installments include both the interest payable each year calculated at 15% of the current outstanding loan and the repayment of a fraction of the loan (amortized loan).

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Example 4.7

Suppose that you took a mortgage of $100,000 on a house to be paid back in 10 equal annual payments (r=6%). If you decided to clear the mortgage after 8 years, how much would you need to pay on top of the 8^{th} installment?

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Definition 4.8

A perpetuity is an infinite sequence of equal payments due at equal time intervals.

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Proposition 4.9

For discrete annual compounding with rate r and payments of C every year, the present value of a perpetuity is

C/r.

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Definition 4.10

We talk about continuous compounding at rate r provided

$V(t)=V(0)e^{rt}$ for all $t\geq 0$.

e^{rt} is called the growth factor,

e^{-rt} is called the discount factor.

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Remark 4.11

Under continuous compounding, the rate of growth of the wealth is proportional to the wealth:

$V'(t)=rV(t)$.

Example 4.12

- Let r=0.1. Find the value of $100 after 1 year with continuous compounding.
- How long does it take to double a capital attracting interest at 6% c.c.?
- What is r if a deposit subject to continuous compounding is doubled after 10 years?

Example 4.12 (continued)

- How long will it take to earn $1 if r=0.1 (c.c.) and V(0)=$1 million?
- Pay $1000 every year after 1, 2, and 3 years. What is the present value of this payment stream? Use continuous compounding at 25%.

Definition 4.13

- Two compounding methods are called equivalent if the corresponding growth factors over a period of one year are the same.
- If one of the growth factors is bigger, then that method is called preferable.

Example 4.14

- What is the equivalent continuous rate for 10% semiannual compounding?
- What is the equivalent quarterly rate for 8% continuous compounding?

Definition 4.15

For a given compounding method, the effective rate r_e is the rate for annual compounding equivalent to that method.

Example 4.16

- What is the effective rate for semiannual compounding with r=10%?
- What is the effective rate for continuous compounding with r=10%?

Definition 4.17

A zero-coupon bond involves a single payment, and the issuing institution promises to exchange the bond for its face value (principal value) at a given maturity date.

Example 4.18

- Suppose a bond has face value F=100 and matures in 1 year. If r=12% (a.c.), find the present value of the bond.
- Find the interest rates for annual, semiannual, and continuous compounding implied by a unit bond with maturity 1 and value 0.9455 after half a year.

Definition 4.19

A coupon bond promises a sequence of payments, consisting of the face value paid at maturity and coupons paid regularly, the last coupon being due at maturity.

Example 4.20

Consider a bond with F=100, T=5, C=10 paid annually, r=0.12 continuously compounded. Find the value of this bond at times 0, 1, and 4.

Proposition 4.21

For coupons paid annually and continuous compounding with constant rate r, the price of a bond with coupon value C, face value F, and maturity T years is

$$C(1-e^{-rT})/(e^{r}-1)+Fe^{-rT}.$$

Definition 4.22

- Assuming that coupons are paid annually, i=C/F is called the coupon rate.
- If the price of a bond is equal to its face value, we say the bond sells at par.
- The coupon rate that causes the bond to sell at par is called the par yield.

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Proposition 4.23

Assume that coupons are paid annually and interest rates are constant. Then the par yield is equal to the effective rate.

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Definition 4.24

The bond yield is the discount rate that, when applied to all cash flows, gives a bond price equal to its market price.

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Example 4.25

Suppose a 2-year Treasury bond with F=100 provides coupons at rate of 6% p.a. semiannually.

Maturity (years)	Treasury zero rate (%) c.c.
0.5	5.0
1.0	5.8
1.5	6.4
2.0	6.8

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Example 4.26

In this example we discuss the most popular approach to calculate Treasury zero rates from the prices of Treasury bonds, the bootstrap method.

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Example 4.26 (continued)

Bond principal ($)	Time to maturity (years)	Annual coupon ($)	Bond price ($)
100	0.25	0	97.5
100	0.50	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0
100	2.00	12	101.6

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Definition 4.27

The forward rate is the future zero rate implied by today's term structure of zero interest rates.

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Example 4.28

Find the forward rates for the nth year (% p.a.).

Year (n)	Zero rate for an n-year investment (% p.a.)
1	3.0
2	4.0
3	4.6
4	5.0
5	5.3

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Proposition 4.29

Assume R_1 and R_2 are the zero rates for maturities T_1 and T_2. Then the forward rate R_F between T_1 and T_2 is given by

$$R_F=(R_2T_2-R_1T_1)/(T_2-T_1).$$

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Definition 4.30

A forward rate agreement (FRA) is an agreement that a certain rate will apply to a certain principal during a certain future period.

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Example 4.31

Suppose company X enters into an FRA with Y that specifies that it will receive a fixed rate of R_K=4% on a principal of L=1 million for a 3-month period starting in 3 years. The actual 3-month LIBOR proves to be R_M=4.5%. Find the cash flow to Y.

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Example 4.32

Suppose rates are as Example 4.28. Consider an FRA where we will receive R_K=6% (annual compounding) on L=1 million between times 1 and 2. Note R_F=5% is the forward rate calculated today. Find the present value of the FRA.

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Definition 4.33

The duration of a bond with price B and yield y that provides cash flow c_i at time t_i, $1 \le i \le n$, is defined by

$$D=\sum_{i=1}^{n} t_i c_i e^{-y t_i}/B$$

Example 4.34

Consider a 3-year 10% coupon bond (paid semiannually) with F=100, y=0.12 cc. Find the duration of the bond.

Remark 4.35

- A zero-coupon bond has duration t_1=T
- Duration is a measure of how long on average the holder has to wait before receiving cash payments
- D is a convex combination of payment times
- Express ΔB in terms of D

Example 4.36

Consider a 3-year 10% coupon bond (paid semiannually) with F=100, y=0.12 cc. Find the new bond price if the yield increases by ten basis points.