Chapter 11

Binomial Trees



Remark 11.1

We let $T > 0$ and write $S_n = S(nT)$ for integers n. We assume S_{n+1} is either $S_n u$ with probability p or $S_n d$ with probability $1 - p$, where

$$0 < d < 1 < u \text{ and } 0 < p < 1$$

("stock price follows random walk").



Example 11.2

Suppose $S_0 = 20, u = 1.1, d = 0.9$. Consider a European call on the stock with strike price $K = 21$ and maturity $T = \frac{1}{4}, r = 0.12$. We seek a risk-less portfolio in stock and option.



Definition 11.3

By a European derivative security or contingent claim with stock S as the underlying asset we mean a random variable of the form $f(S(T))$, where f is a given function, called the payoff.



Example 11.4

- call: $f(S) = (S-K)^+$
- put: $f(S) = (K-S)^+$
- long forward: $f(S) = S - K$



Theorem 11.5

In the one-step binomial tree model,

$$f = \mathbb{E}_*\big(f(S_1)\big)e^{-rT}$$

("the present value of the contingent claim is equal to the discounted payoff expectation in a risk-neutral world, independent of p"). Here $p_* = \frac{e^{rT}-d}{u-d}$.



Example 11.6

Suppose $S_0 = 20$, $u = 1.1$, $d = 0.9$, $K = 21$, $r = 0.12$, $T = 0.25$.

a) In the one-step BM, find c.

b) Find also p.

c) Show that PCP holds.



Theorem 11.7

In the n-step binomial tree model,

$$f = \mathbb{E}_*\big(f(S_n)\big)e^{-nrT}.$$



Example 11.8

a) Example 11.6 with $n = 2$

b) 2-year European put, $K = 52$, $S_0 = 50$, $r = 0.05$, $T = 1$, $u = 1.2$, $d = 0.8$

c) American put (as above)



Theorem 11.9

(Cox-Ross-Rubinstein formula)

In the binomial model, the price of a European call and put with strike price K to be exercised after N time steps is

$$c = S_0[1 - \Phi(m-1, N, q)] - Ke^{-rNT}[1 - \Phi(m-1, N, p_*)]$$



Theorem 11.9 (continued)

and

$$p = S_0\Phi(m-1, N, q) - Ke^{-rNT}\Phi(m-1, N, p_*),$$

where Φ is the cdf of the binomial distribution, $q = p_* u e^{-rT}$, and m is the smallest nonnegative integer with $S_0 u^m d^{N-m} > K$.



Remark 11.10

The expected return μ and the volatility σ of a stock price will be defined in such a way that

$$\mathbb{E}(S_1) = S_0 e^{\mu T} \text{ and } \mathbb{V}(S_1) = S_0^2 \sigma^2 T$$

for small T in a one-step BM. Then

$$u = e^{\sigma\sqrt{T}} \text{ and } d = e^{-\sigma\sqrt{T}}.$$



Example 11.11

Example 11.8(c), where we assume that volatility is 30%

$$p = \frac{a-d}{u-d}.$$



Example 11.12

For stocks paying a continuous dividend yield, use formulas as above except

$$a = e^{(r-q)T}.$$

E.g., stock index with $S_0 = 810$, $\sigma = 0.2$, $q = 0.02$, $r = 0.05$, $K = 800$, $T = 0.25$, $N = 2$, European call.



Example 11.13

Foreign currencies can be regarded as an asset providing a yield r_f, use

$$a = e^{(r-r_f)T}.$$

E.g., AUD is currently worth 0.61 USD and this exchange rate has a volatility of 12%, $r_f = 0.07$, $r = 0.05$. Value a 3-month American call with strike price of 0.6 using a three step tree.



Example 11.14

Options on futures: Futures price should have an expected growth rate of zero. Use previous formulas with

$$a = 1.$$

