Chapter 13

The Black-Scholes-Merton Model



Definition 13.1

A nonnegative random variable X is said to have a lognormal distribution with parameters μ and σ if

$$\ln(X) \sim N(\mu, \sigma).$$



Remark 13.2

If S is a stock price following geometric Brownian motion, then S_T has a lognormal distribution, namely

$$\ln(S_T) \sim N\left(\ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right)$$



Example 13.3

Consider a stock with an initial price of \$40, an expected return of 16% (pa), and a volatility of 20% (pa). Find a 95%-confidence interval for $S_{1/2}$.





Theorem 13.4

If X has a lognormal distribution with parameters μ and σ, then

$$\mathbb{E}(X) = e^{\mu+\frac{\sigma^2}{2}}$$

and

$$\mathbb{V}(X) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1).$$



Remark 13.5

If S is a stock price following geometric Brownian motion, then

$$\mathbb{E}(S_T) = S_0 e^{\mu T}$$

and

$$\mathbb{V}(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1).$$



Example 13.6

Consider a stock with current price of $20, an expected return of 20% (pa), and a volatility of 40% (pa). Find the expected value and the variance of the stock price in one year.



Remark 13.7

Here we discuss how to estimate volatility.



Theorem 13.8

(Black-Scholes-Merton PDE)

If $\mathrm{d}S = \mu S\mathrm{d}t + \sigma S\mathrm{d}W$ and f is the price of a call, then

$$f_t + rSf_S + \frac{\sigma^2 S^2}{2} f_{SS} = rf.$$



Remark 13.9

(BSM) has many solutions, but we are looking for a solution that satisfies the boundary condition

$f = (S - K)^+$ when $t = T$

for a European call, or

$f = (K - S)^+$ when $t = T$

for a European put.



Remark 13.10

The portfolio from the proof of Theorem 13.8 is not permanently riskless, only during Δt. To keep the portfolio riskless, frequent adjustments are to be made.



Remark 13.11

Any f that satisfies (BSM) is called a price of a tradeable derivative.



Example 13.12

a) $f(S,t) = S - Ke^{-r(T-t)}$ is a price of a tradeable derivative.

b) $f(S,t) = e^{S}$ is not a ptd.

c) $f(S,t) = \dfrac{e^{(\sigma^2-2r)(T-t)}}{S}$ is a ptd.

Theorem 13.13

(Black-Scholes pricing formulas)

The prices at time 0 of a European call and put on a non-dividend-paying stock are

$$c = S_0 N(d_+) - Ke^{-rT} N(d_-)$$

and

$$p = Ke^{-rT} N(-d_-) - S_0 N(-d_+).$$

Theorem 13.13 (continued)

Here, N is the cdf of the standard normal distribution and

$$d_{\pm} = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}.$$

Remark 13.14

- Let $S_0 \to \infty$
- Let $\sigma \to 0$
- We can use the polynomial approximation providing 6-decimal-place accuracy

$N(x) = 1 - N'(x)(a_1 k + a_2 k^2 + a_3 k^3 + a_4 k^4 + a_5 k^5)$, $x \geq 0$,
$k = \dfrac{1}{1 + x\eta}$, $\eta = 0.2316419$, $a_1 = 0.319381530$,
$a_2 = -0.356563782$, $a_3 = 1.781477937$,
$a_4 = -1.821255978$, $a_5 = 1.330274429$.

Example 13.15

$T = 0.5, S_0 = 42, K = 40, r = 0.1, \sigma = 0.2$. Find the prices of a European call and a European put.




Remark 13.16

We can also use risk-neutral valuation to prove BSPF:

- Assume that the expected return from the stock price is the risk-free rate.
- Calculate the expected payoff from the option.
- Discount at the risk-free rate.



Theorem 13.17

Let X be lognormally distributed with $\ln(X) \sim N(m, w)$. Then for $K > 0$,

$$\mathbb{E}((X-K)^+) = \mathbb{E}(X)N(d_+) - KN(d_-),$$

where

$$d_{\pm} = \frac{\ln\left(\frac{\mathbb{E}(X)}{K}\right) \pm \frac{w^2}{2}}{w}.$$



Chapter 14

Options on Stock Indices, Currencies, and Futures

