Chapter 1
Probability Background

1.1 Probability Spaces

Definition 1.1. Let Ω be a set. We say that a collection $\mathcal{F}$ of subsets of Ω is a σ-*algebra* provided

(i) $\emptyset \in \mathcal{F}$;

(ii) if $A \in \mathcal{F}$, then $A^{\mathrm{c}} \in \mathcal{F}$;

(iii) if $A_n \in \mathcal{F}$ for all $n \in \mathbb{N}$, then $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{F}$.

Example 1.2. Let $\Omega = \{UUU, UUD, UDU, UDD, DUU, DUD, DDU, DDD\}$ in the BAPM (Binomial Asset Pricing Model) with $N = 3$.

Definition 1.3. Let Ω be a set and let $\mathcal{F}$ be a σ-algebra of subsets of Ω. We call $\mathbb{P} : \mathcal{F} \to [0, 1]$ a *probability measure* on $\mathcal{F}$ provided

(i) $\mathbb{P}(\Omega) = 1$;

(ii) if $A_n \in \mathcal{F}$ for all $n \in \mathbb{N}$ are disjoint, then $\mathbb{P}\left(\bigcup\limits_{n\in\mathbb{N}} A_n\right) = \sum\limits_{n\in\mathbb{N}} \mathbb{P}(A_n)$.

The measure is called complete if

(iii) $A \in \mathcal{F}$, $B \subset A$, $\mathbb{P}(A) = 0$ imply $B \in \mathcal{F}$, $\mathbb{P}(B) = 0$.

Example 1.4. For $A \in \mathcal{P}(\Omega)$ from Example 1.2, define $\mathbb{P}(A) = \sum\limits_{\omega\in A} \mathbb{P}(\{\omega\})$.

Definition 1.5. A triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space* or *Kolmogorov triple* provided

(i) Ω is any set;

(ii) $\mathcal{F}$ is a σ-algebra of subsets of Ω;

(iii) $\mathbb{P}$ is a probability measure on $\mathcal{F}$.

Remark 1.6. (i) A point $\omega \in \Omega$ is called *sample point*;

(ii) a set $A \in \mathcal{F}$ is called an *event*;

(iii) $\mathbb{P}(A)$ is called the *probability* of the event A;

(iv) a property which is true except for an event of probability zero is said to hold *almost surely* (abbreviated by a.s.);

(v) by $\mathcal{B} = \mathcal{B}(\mathbb{R}^n)$ we denote the collection of *Borel subsets* of $\mathbb{R}^n$, which is the smallest σ-algebra of subsets of $\mathbb{R}^n$ containing all open sets.

1.2 Random Variables

Definition 1.7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A mapping $X : \Omega \to \mathbb{R}^n$ is called an n-dimensional *random variable* if

$$X^{-1}(B) \in \mathcal{F} \quad \text{for all} \quad B \in \mathcal{B}.$$

We also say that X is *$\mathcal{F}$-measurable*.

Lemma 1.8. *Let $X : \Omega \to \mathbb{R}^n$ be a mapping. Then*

$$\sigma(X) := \{X^{-1}(B) : \ B \in \mathcal{B}\}$$

is a σ-algebra, called the σ-algebra generated by X. This is the smallest σ-algebra of Ω with respect to which X is measurable.

Example 1.9. Let $S_0 = 4$, $u = 2$, $d = 1/2$ in the BAPM. with $N = 3$. Consider S_2, the price of the stock at time 2.

1.3 Lebesgue Theory

Definition 1.10. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let X be a random variable.

(i) If X is an *indicator* random variable, we define

$$\int_\Omega X \mathrm{d}\mathbb{P} := \mathbb{P}(A), \quad \text{where} \quad X = \chi_A;$$

(ii) If X is a *simple* random variable, we define

$$\int_\Omega X \mathrm{d}\mathbb{P} := \sum_{i=1}^{k} a_i \int_\Omega \chi_{A_i} \mathrm{d}\mathbb{P}, \quad \text{where} \quad X = \sum_{i=1}^{k} a_i \chi_{A_i};$$

(iii) if X is a *nonnegative* random variable, we define

$$\int_\Omega X \mathrm{d}\mathbb{P} := \sup \left\{ \int_\Omega Y \mathrm{d}\mathbb{P} : \ Y \leq X, \ Y \text{ simple} \right\};$$

(iv) if X is *any* random variable, we define

$$\int_\Omega X \mathrm{d}\mathbb{P} := \int_\Omega X^+ \mathrm{d}\mathbb{P} - \int_\Omega X^- \mathrm{d}\mathbb{P},$$

provided at least one of the integrals on the right is finite. Here,

$$X^+ := \frac{|X| + X}{2} \quad \text{and} \quad X^- := \frac{|X| - X}{2}.$$

Definition 1.11. We call

$$\mathbb{E}(X) := \int_\Omega X \mathrm{d}\mathbb{P}$$

the *expected value* of X, while

$$\mathbb{V}(X) := \mathbb{E}(|X - \mathbb{E}(X)|^2).$$

denotes the *variance* of X. Moreover,

$$\mathbb{C}\text{ov}(X, Y) := \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y)))$$

and

$$\rho(X, Y) := \frac{\mathbb{C}\text{ov}(X, Y)}{\sqrt{\mathbb{V}(X)\mathbb{V}(Y)}} \quad \text{if} \quad \mathbb{V}(X)\mathbb{V}(Y) \neq 0$$

denote the *covariance* and *correlation coefficient* of X and Y, respectively. If $\rho(X, Y) = 0$, then X and Y are called *uncorrelated*.

Theorem 1.12 (Linearity of Expectation). *We have*

$$\mathbb{E}(\alpha X) = \alpha \mathbb{E}(X) \quad \textit{and} \quad \mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y).$$

Theorem 1.13 (Continuity of Expectation). *Suppose X_n are random variables with*

$$X_n \to X \quad \textit{a.s.,} \quad n \to \infty.$$

(i) Fatou's lemma: If $X_n \geq Y$ a.s. for all $n \in \mathbb{N}$, where $\mathbb{E}(|Y|) < \infty$,

$$\mathbb{E}\left(\liminf_{n\to\infty} X_n\right) \leq \liminf_{n\to\infty} \mathbb{E}(X_n).$$

(ii) Monotone convergence: If $0 \leq X_n \leq X_{n+1}$ a.s. for all $n \in \mathbb{N}$, then

$$\mathbb{E}(X_n) \to \mathbb{E}(X) \quad \textit{as} \quad n \to \infty.$$

(iii) Dominated convergence: If $|X_n| \leq Y$ a.s. for all $n \in \mathbb{N}$, where $\mathbb{E}(Y) < \infty$, then

$$\mathbb{E}(X_n) \to \mathbb{E}(X) \quad \textit{as} \quad n \to \infty.$$

(iv) *Bounded convergence: If* $|X_n| \le c$ *a.s. for all* $n \in \mathbb{N}$, *where* $c \in \mathbb{R}$, *then*

$$\mathbb{E}(X_n) \to \mathbb{E}(X) \quad \text{as} \quad n \to \infty.$$

Theorem 1.14 (Inequalities). *The following inequalities hold:*

(i) *Hölder:*

$$\mathbb{E}(|XY|) \le (\mathbb{E}(|X|^p))^{1/p}(\mathbb{E}(|Y|^q))^{1/q}, \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

(ii) *Cauchy–Schwarz:*

$$\mathbb{E}(|XY|) \le \sqrt{\mathbb{E}(X^2)\mathbb{E}(Y^2)},$$

in particular

$$|\operatorname{\mathbb{C}ov}(X,Y)| \le \sqrt{\mathbb{V}(X)\mathbb{V}(Y)} \quad \text{and} \quad |\rho(X,Y)| \le 1.$$

(iii) *Minkowski:*

$$(\mathbb{E}(|X+Y)|^p))^{1/p} \le (\mathbb{E}(|X|^p))^{1/p} + (\mathbb{E}(|Y|^p))^{1/p}.$$

(iv) *Markov: If* $X \ge 0$ *a.s., then*

$$\mathbb{P}(X \ge c) \le \frac{\mathbb{E}(X)}{c} \quad \text{for} \quad c > 0.$$

(v) *Čebyshev:*

$$\mathbb{P}(|X| \ge c) \le \frac{\mathbb{E}(X^2)}{c^2} \quad \text{for} \quad c > 0.$$

(vi) *Jensen: If* $\phi : \mathbb{R} \to \mathbb{R}$ *is convex, then*

$$\mathbb{E}(\phi(X)) \ge \phi(\mathbb{E}(X)).$$

1.4 Independence

Definition 1.15. Let $\mathcal{F}$ be a σ-algebra. Two events $A, B \in \mathcal{F}$ are called *independent* provided

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

Furthermore, if $\mathbb{P}(B) > 0$, then we define

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Example 1.16. In the BAPM with $N = 2$, $A = \{UU, UD\}$ and $B = \{UD, DU\}$ are independent iff $p = 1/2$.

Definition 1.17. Let $\mathcal{F}$ be a σ-algebra and let $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ be sub-σ-algebras of $\mathcal{F}$. Then $\mathcal{G}$ and $\mathcal{H}$ are called *independent* provided

$$A, B \text{ are independent} \qquad \text{for all} \qquad A \in \mathcal{G} \text{ and } B \in \mathcal{H}.$$

Definition 1.18. Two random variables X and Y are called *independent* provided $\sigma(X)$ and $\sigma(Y)$ are independent.

Example 1.19. In the BAPM with $N = 2$, consider S_1 and S_2.

Theorem 1.20. *If X and Y are independent, then they are uncorrelated. But the converse is not true in general.*

Theorem 1.21. *If X and Y are independent and $g, h : \mathbb{R} \to \mathbb{R}$ are Borel measurable, then*

(i) $g(X)$ *and* $h(Y)$ *are independent;*

(ii) $\mathbb{E}(g(X)h(Y)) = \mathbb{E}(g(X))\mathbb{E}(h(Y))$.

1.5 Change of Measure

Definition 1.22. Let $\mathbb{P}$ and $\tilde{\mathbb{P}}$ be two probability measures on $(\Omega, \mathcal{F})$. We say that $\tilde{\mathbb{P}}$ is *absolutely continuous* with respect to $\mathbb{P}$ provided

$$\mathbb{P}(A) = 0, \quad A \in \mathcal{F} \quad \text{implies} \quad \tilde{\mathbb{P}}(A) = 0.$$

In this case we write $\tilde{\mathbb{P}} \ll \mathbb{P}$. If both $\tilde{\mathbb{P}} \ll \mathbb{P}$ and $\mathbb{P} \ll \tilde{\mathbb{P}}$, then we say that $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are *equivalent* measures and write $\mathbb{P} \sim \tilde{\mathbb{P}}$.

Theorem 1.23 (Radon–Nikodým). *Let $\mathbb{P}$ and $\tilde{\mathbb{P}}$ be two probability measures on $(\Omega, \mathcal{F})$. If $\tilde{\mathbb{P}} \ll \mathbb{P}$, then there exists a nonnegative random variable Z with*

$$\tilde{\mathbb{P}}(A) = \int_A Z \mathrm{d}\mathbb{P} \quad \textit{for all} \quad A \in \mathcal{F}.$$