Chapter 2
Conditional Expectation

2.1 Definition and Properties of Conditional Expectation

Definition 2.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Suppose $X : \Omega \to \mathbb{R}$ is a random variable with $\mathbb{E}(|X|) < \infty$. Then the *conditional expectation* of X with respect to $\mathcal{G}$ is defined to be $\mathbb{E}(X|\mathcal{G}) := Y$, where Y is any random variable satisfying

(i) $\mathbb{E}(|Y|) < \infty$;
(ii) Y is $\mathcal{G}$-measurable;
(iii) partial averaging property:

$$\int_A Y \mathrm{d}\mathbb{P} = \int_A X \mathrm{d}\mathbb{P} \quad \text{for all} \quad A \in \mathcal{G}.$$

Moreover, if Z is a random variable, we write $\mathbb{E}(X|Z) = \mathbb{E}(X|\sigma(Z))$.

Example 2.2. In the BAPM with $N = 2$, for $\mathcal{F}_1 = \{\emptyset, A_U, A_D, \Omega\}$, find $\mathbb{E}(S_2|\mathcal{F}_1)$.

Theorem 2.3 (Existence and Uniqueness of Conditional Expectation). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Suppose $X : \Omega \to \mathbb{R}$ is a random variable with $\mathbb{E}(|X|) < \infty$. Then the conditional expectation $\mathbb{E}(X|\mathcal{G})$ exists and is unique up to $\mathcal{G}$-measurable sets of probability zero.*

Theorem 2.4 (Properties of Conditional Expectation). *We have:*

(i) Conditional mean formula: $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$;
(ii) $\mathbb{E}(X|\mathcal{G}) = X$ *a.s. if* X *is* $\mathcal{G}$*-measurable;*
(iii) Linearity: $\mathbb{E}(a_1X_1 + a_2X_2|\mathcal{G}) = a_1\mathbb{E}(X_1|\mathcal{G}) + a_2\mathbb{E}(X_2|\mathcal{G})$;
(iv) Positivity: $\mathbb{E}(X|\mathcal{G}) \geq 0$ *a.s. if* $X \geq 0$ *a.s.;*
(v) Tower property: $\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(X|\mathcal{H})$ *if* $\mathcal{H}$ *is a sub-*σ*-algebra of* $\mathcal{G}$*;*
(vi) Tower property: $\mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G}) = \mathbb{E}(X|\mathcal{H})$ *if* $\mathcal{H}$ *is a sub-*σ*-algebra of* $\mathcal{G}$*;*
(vii) Taking out what is known: $\mathbb{E}(XZ|\mathcal{G}) = Z\mathbb{E}(X|\mathcal{G})$ *if* Z *is* $\mathcal{G}$*-measurable and bounded;*
(viii) Role of independence: $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ *if* X *is independent of* $\mathcal{G}$*;*
(ix) Conditional Jensen inequality: $\phi(\mathbb{E}(X|\mathcal{G})) \leq \mathbb{E}(\phi(X)|\mathcal{G})$ *if* $\phi : \mathbb{R} \to \mathbb{R}$ *is convex and* $\mathbb{E}(|\phi(X)|) < \infty$.

2.2 Martingales

Definition 2.5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(i) A sequence $\mathbb{F} := \{\mathcal{F}_n : n \in \mathbb{N}_0\}$ of sub-σ-algebras of $\mathcal{F}$ is called a *filtration* provided

$$\mathcal{F}_n \subset \mathcal{F}_{n+1} \quad \text{for all} \quad n \in \mathbb{N}_0,$$

and then $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ is called a *stochastic basis* or *filtered probability space*.

(ii) A sequence of random variables $X := \{X_n : n \in \mathbb{N}_0\}$ (defined on Ω and $\mathcal{F}$-measurable) is called a *discrete stochastic process*;

(iii) The stochastic process X is said to be *adapted* to the filtration $\mathbb{F}$ provided

$$X_n \quad \text{is } \mathcal{F}_n\text{-measurable} \qquad \text{for all} \quad n \in \mathbb{N}_0.$$

Definition 2.6. Let $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ be a filtered probability space. Let M be a discrete stochastic process that is adapted to $\mathbb{F}$ such that $\mathbb{E}(|M_n|) < \infty$ for all $n \in \mathbb{N}$. Then M is called a

(i) *martingale* if

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = M_n \text{ a.s.} \quad \text{for all} \quad n \in \mathbb{N}_0;$$

(ii) *supermartingale* if

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) \leq M_n \text{ a.s.} \quad \text{for all} \quad n \in \mathbb{N}_0;$$

(iii) *submartingale* if

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) \geq M_n \text{ a.s.} \quad \text{for all} \quad n \in \mathbb{N}_0.$$

Example 2.7. In the BAPM, S is a martingale, supermartingale, or submartingale provided $pu + qd = 1$, $pu + qd \leq 1$, or $pu + qd \geq 1$, respectively.

Theorem 2.8. *If $\{M_n, \mathcal{F}_n\}$ is a martingale, then*

$$\mathbb{E}(M_m|\mathcal{F}_n) = M_n \quad \textit{for all} \quad 0 \leq n \leq m.$$

Theorem 2.9. *The expectation of a martingale is constant over time.*