Chapter 3

European Derivative Securities in Discrete Time

3.1 Risk-neutral Measures

Definition 3.1. Define the *discount factor* $\beta_k = (1+r)^{-k}$. If there exists $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ with $\tilde{\mathbb{P}} \sim \mathbb{P}$ such that $\{\beta_k S_k, \mathcal{F}_k\}$ is a martingale under $\tilde{\mathbb{P}}$, then we say that $\tilde{\mathbb{P}}$ is a *risk-neutral measure*. The expectation under $\tilde{\mathbb{P}}$ is denoted by $\tilde{\mathbb{E}}$.

Theorem 3.2. *In the BAPM, let*

$$\tilde{p} = \frac{(1+r)-d}{u-d} \quad \textit{and} \quad \tilde{q} = \frac{u-(1+r)}{u-d}$$

and define a probability measure $\tilde{\mathbb{P}}$ on Ω by

$$\tilde{\mathbb{P}}(\omega) := \tilde{p}^{\textit{number of U in } \omega} \tilde{q}^{\textit{number of D in } \omega}.$$

Then $\tilde{\mathbb{P}}$ is a risk-neutral measure.

Remark 3.3. We may derive the "risk-neutral" probabilities $\tilde{p}$ and $\tilde{q}$ from Theorem 3.2 by "replicating an option with stocks".

Definition 3.4. A *portfolio process* φ is a process adapted to a filtration. For a given constant *initial wealth* X_0 and a portfolio process φ, we define the *wealth process* X recursively by

$$X_{k+1} = \varphi_k S_{k+1} + (1+r)(X_k - \varphi_k S_k), \quad 0 \le k \le N-1.$$

Theorem 3.5. *Under $\tilde{\mathbb{P}}$, the discounted wealth process βX is a martingale.*

Corollary 3.6. *The discounted wealth process satisfies*

$$\tilde{\mathbb{E}}(\beta_n X_n) = X_0 \quad \textit{for all} \quad 0 \le n \le N.$$

Definition 3.7. A *simple European derivative security* (sEds) (or *contingent claim*) with expiration time $N \in \mathbb{N}$ is an $\mathcal{F}_N$-measurable random variable.

Definition 3.8. An sEds V_N is said to be *hedgeable* (or *attainable*) if there exists a constant X_0 and a portfolio process φ such that the wealth process X given in Definition 3.4 satisfies

$$X_N = V_N.$$

In this case, for $0 \le k \le N$, we call X_k the *APT value* (Arbitrage Pricing Theory) at time k of V_N.

Theorem 3.9. *If an sEds V_N is hedgeable, then the APT value at time k of V_N is*

$$V_k := \beta_k^{-1}\tilde{\mathbb{E}}(\beta_N V_N | \mathcal{F}_k), \quad 0 \le k \le N-1,$$

in particular

$$V_0 = \tilde{\mathbb{E}}(\beta_N V_N).$$

Theorem 3.10. *Let V_N be any sEds and define the value process V as in Theorem 3.9. Then βV is a martingale under $\tilde{\mathbb{P}}$.*

Definition 3.11. A model in which any sEds is hedgeable is called *complete*. Otherwise, the model is called *incomplete*.

Theorem 3.12. *The BAPM is complete.*

Example 3.13. Consider a European call with strike price $K = 5$ and expiration time $N = 2$ in the BAPM with $p = 1/3$, $q = 2/3$, $r = 1/4$, $S_0 = 4$, $u = 2$, $d = 1/2$.

3.2 Change of Measure

Theorem 3.14. *Assume $\mathbb{P} \sim \tilde{\mathbb{P}}$ and let Z be the Radon–Nikodým derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$. Then*

(i) $\mathbb{P}(Z > 0) = 1$*;*

(ii) $\mathbb{E}(Z) = 1$*;*

(iii) $\tilde{\mathbb{E}}(Y) = \mathbb{E}(YZ)$ *for any random variable Y;*

(iv) $\tilde{\mathbb{P}}(1/Z > 0) = 1$*;*

(v) $\tilde{\mathbb{E}}(1/Z) = 1$*;*

(vi) $\mathbb{E}(Y) = \tilde{\mathbb{E}}(Y/Z)$ *for any random variable Y.*

Example 3.15. Consider the BAPM with $p = 1/3$, $q = 2/3$, $\tilde{p} = \tilde{q} = 1/2$, $r = 1/4$.

Theorem 3.16. *Assume $\mathbb{P} \sim \tilde{\mathbb{P}}$ and let Z be the Radon–Nikodým derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$. Define $Z_n = \mathbb{E}(Z|\mathcal{F}_n)$. Then*

(i) $\{Z_n\}$ *is a martingale;*

(ii) $\tilde{\mathbb{E}}(X) = \mathbb{E}(XZ_k)$ *if X is $\mathcal{F}_k$-measurable;*

(iii) $\tilde{\mathbb{E}}(X|\mathcal{F}_j) = \frac{1}{Z_j}\mathbb{E}(XZ_k|\mathcal{F}_j)$, $0 \le j \le k$, *if X is $\mathcal{F}_k$-measurable and bounded.*

Definition 3.17. The *state price density process* is given by $\zeta_k = \beta_k Z_k$.

Theorem 3.18. *For an sEds V_N, the value process V from Theorem 3.9 satisfies*

(i) $V_0 = \mathbb{E}(\zeta_N V_N)$;

(ii) $V_k = \zeta_k^{-1} \mathbb{E}(\zeta_N V_N | \mathcal{F}_k)$ *for all* $0 \leq k \leq N$;

(iii) ζV *is a martingale under* $\mathbb{P}$.

Example 3.19. Revisit Example 3.13 using Theorem 3.18.

3.3 Fundamental Theorem of Asset Pricing

Definition 3.20. Let $X_0 = 0$. If there exists a portfolio process such that the wealth process satisfies

$$\mathbb{P}(X_N \geq 0) = 1 \quad \text{and} \quad \mathbb{P}(X_N > 0) > 0,$$

then we say that an *arbitrage opportunity* exists. A market is called *arbitrage free* if there is no arbitrage opportunity.

Theorem 3.21 (No-arbitrage Theorem). *A market is arbitrage free if and only if there exists a risk-neutral measure.*

Theorem 3.22 (Completeness Theorem). *An arbitrage-free market is complete if and only if there exists a unique risk-neutral measure.*

Theorem 3.23 (Fundamental Theorem of Asset Pricing). *In an arbitrage-free complete market, there exists a unique risk-neutral measure.*