Chapter 9
Exotic Options

9.1 Maximum of Brownian Motion

Definition 9.1. The *maximum to date* for a Brownian motion W is defined by

$$M(t) = \max_{0 \le s \le t} W(s).$$

Lemma 9.2 (Reflection Equality). *If W is Brownian motion and M is its maximum to date, then*

$$\mathbb{P}(M(t) \ge m, W(t) \le w) = \mathbb{P}(W(t) \ge 2m - w) \quad \textit{for} \quad w \le m, \quad m > 0.$$

Theorem 9.3. *For $t > 0$, the joint density of $(M(t), W(t))$ is*

$$f_{M(t),W(t)}(m, w) = \frac{2(2m - w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}} \quad \textit{for} \quad w \le m, \quad m > 0.$$

Definition 9.4. Let $\tilde{W}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$. We define the *Brownian motion with a drift* α under $\tilde{\mathbb{P}}$ by

$$\hat{W}(t) = \alpha t + \tilde{W}(t) \quad \text{for} \quad 0 \le t \le T.$$

Theorem 9.5. *The joint density under $\tilde{\mathbb{P}}$ of $(\hat{M}(T), \hat{W}(T))$ is*

$$\tilde{f}_{\hat{M}(T),\hat{W}(T)}(m, w) = \frac{2(2m - w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{\alpha^2 T}{2} - \frac{(2m-w)^2}{2T}} \quad \textit{for} \quad w \le m, \quad m > 0.$$

Theorem 9.6. *We have*

$$\tilde{\mathbb{P}}(\hat{M}(T) \le m) = N\left(\frac{m - \alpha T}{\sqrt{T}}\right) - e^{2\alpha m} N\left(\frac{-m - \alpha T}{\sqrt{T}}\right) \quad \textit{for} \quad m \ge 0,$$

and the density of the random variable $\hat{M}(T)$ under $\tilde{\mathbb{P}}$ is

$$\tilde{f}_{\hat{M}(T)}(m) = \sqrt{\frac{2}{\pi T}} e^{-\frac{(m-\alpha T)^2}{2T}} - 2\alpha e^{2\alpha m} N\left(\frac{-m - \alpha T}{\sqrt{T}}\right) \quad \textit{for} \quad m \ge 0.$$

9.2 Knock-out Barrier Options

Definition 9.7. An *up-and-out European call* with strike price K and *up-and-out barrier* B pays off $(S(T)-K)^+$ if $\max_{0\le t\le T} S(t)\le B$ and 0 otherwise.

Theorem 9.8. *Assume r and σ are constant. The price of an up-and-out European call at time 0 is*

$$\begin{aligned} V(0) \;=\; & S(0)\left\{N\left(\delta_+\left(T,\frac{S(0)}{K}\right)\right)-N\left(\delta_+\left(T,\frac{S(0)}{B}\right)\right)\right\} \\ & -Ke^{-rT}\left\{N\left(\delta_-\left(T,\frac{S(0)}{K}\right)\right)-N\left(\delta_-\left(T,\frac{S(0)}{B}\right)\right)\right\} \\ & -B\left(\frac{S(0)}{B}\right)^{-\frac{2r}{\sigma^2}}\left\{N\left(\delta_+\left(T,\frac{B^2}{S(0)K}\right)\right)-N\left(\delta_+\left(T,\frac{B}{S(0)}\right)\right)\right\} \\ & +Ke^{-rT}\left(\frac{S(0)}{B}\right)^{1-\frac{2r}{\sigma^2}}\left\{N\left(\delta_-\left(T,\frac{B^2}{S(0)K}\right)\right)-N\left(\delta_-\left(T,\frac{B}{S(0)}\right)\right)\right\}, \end{aligned}$$

where

$$\delta_\pm(\tau,x)=\frac{\ln(x)+\left(r\pm\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

9.3 Lookback Options

Definition 9.9. A *floating strike lookback option* pays off $\max_{0\le t\le T} S(t)-S(T)$.

Theorem 9.10. *Assume r and σ are constant. The price of a floating strike lookback option at time t is*

$$\begin{aligned} V(t) \;=\; & e^{-r\tau}Y(t)N\left(-\delta_-\left(\tau,\frac{S(t)}{Y(t)}\right)\right) \\ & -\frac{\sigma^2}{2r}\left(\frac{Y(t)}{S(t)}\right)^{\frac{2r}{\sigma^2}}S(t)e^{-r\tau}N\left(-\delta_-\left(\tau,\frac{Y(t)}{S(t)}\right)\right) \\ & +\left(1+\frac{\sigma^2}{2r}\right)S(t)N\left(\delta_+\left(\tau,\frac{S(t)}{Y(t)}\right)\right)-S(t), \end{aligned}$$

where

$$\tau=T-t \quad \textit{and} \quad Y(t)=\max_{0\le u\le t} S(u).$$

In particular,

$$V(0)=S(0)\left\{\left(1-\frac{\sigma^2}{2r}\right)e^{-rT}N\left(\frac{\frac{\sigma^2}{2}-r}{\sigma}\sqrt{T}\right)+\left(1+\frac{\sigma^2}{2r}\right)N\left(\frac{\frac{\sigma^2}{2}+r}{\sigma}\sqrt{T}\right)-1\right\}.$$