1. Prove that if $\mathcal{F}_n \subset \Omega$ are σ-algebras for all $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} \mathcal{F}_n$ is also a σ-algebra.
2. Prove Lemma 1.8 from the Lecture Notes.
3. Consider a TAPM (trinomial asset pricing model), where in addition to U (going up) and D (going down) there is also a possibility of S (stay). Let $N = 2$. Find Ω and a nontrivial σ-algebra.
4. In the TAPM with $N = 2$, find S_1 and S_2 as well as $\sigma(S_1)$ and $\sigma(S_2)$.
5. In the TAPM with $N = 2$, assume the probability of an upward move is 1/4 and so is the probability of a downward move. Find $\mathbb{E}(S_2)$.
6. In the TAPM with $N = 2$, find probabilities that ensure that $\{UU, UD\}$ and $\{UD, DU\}$ are independent.
7. Let A and B be events with $\mathbb{P}(A) = 3/4$ and $\mathbb{P}(B) = 1/3$. Show that $1/12 \le \mathbb{P}(A \cap B) \le 1/3$ and give corresponding bounds for $\mathbb{P}(A \cup B)$.
8. Let A_n for $n \in \mathbb{N}$ be events and prove that
$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} \mathbb{P}(A_i) \quad \text{and} \quad \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \ge 1 - n + \sum_{i=1}^{n} \mathbb{P}(A_i)$$
holds for all $n \in \mathbb{N}$.
9. If two events A and B are independent, prove that A^{c} and B^{c} are independent.
10. Assume that B_n for $n \in \mathbb{N}$ are disjoint events that have as a union the entire sample space. Prove that
$$\mathbb{P}(A) = \sum_{n=1}^{\infty} \mathbb{P}(A|B_n)\mathbb{P}(B_n)$$
holds for any event A.
11. Suppose the random variable X is nonnegative almost surely. Prove that $\mathbb{E}(X) = 0$ iff $X = 0$ almost surely.