12. In the BAPM, find $\mathbb{E}(S_3|S_2)$.
13. In the BAPM, find $\mathbb{E}(S_1|S_2)$.
14. In the TAPM, find $\mathbb{E}(S_2|S_1)$.
15. In the TAPM, find $\mathbb{E}(S_1|S_2)$.
16. Prove Theorem 2.4 (iii) from the Lecture Notes.
17. For the BAPM, show directly that the following properties from Theorem 2.4 hold:
    (a) Linearity with $a_1 = a_2 = 1$, $X_1 = S_2$, $X_2 = S_3$, $\mathcal{G} = \mathcal{F}_1$.
    (b) Taking out what is known with $XZ = S_1 S_2$, $\mathcal{G} = \mathcal{F}_1$.
    (c) Tower property with $X = S_3$, $\mathcal{H} = \mathcal{F}_1$, $\mathcal{G} = \mathcal{F}_2$.
    (d) Independence with $X = S_2/S_1$, $\mathcal{G} = \mathcal{F}_1$.
18. For the TAPM, is S a martingale, submartingale, or supermartingale?
19. Show that a convex function of a martingale is a submartingale.
20. Let M be a martingale and φ an adapted process. Define the discrete-time stochastic integral I by $I_0 = 0$ and

$$I_n = \sum_{j=0}^{n-1} \varphi_j (M_{j+1} - M_j), \quad n \in \mathbb{N}.$$

    Show that I is a martingale.
21. Toss a coin repeatedly. Let $X_j = 1$ if the jth toss is a head and $X_j = -1$ if the jth toss is a tail. Define $M_0 = 0$ and

$$M_n = \sum_{j=1}^{n} X_j, \quad n \in \mathbb{N}.$$

    (a) Show that M is a martingale.
    (b) Let $\sigma > 0$, define

$$N_n = e^{\sigma M_n} \left(\frac{2}{e^{\sigma} + e^{-\sigma}} \right)^n, \quad n \in \mathbb{N}_0,$$

    and show that N is a martingale.