40. Show that the mean of a martingale, a supermartingale, and a submartingale is constant, nonincreasing, and nondecreasing, respectively.

41. Let W be Brownian motion. Find

$$\text{(a) } \mathbb{P}(W(2) \le 0); \quad \text{(b) } \mathbb{P}(W(0) \le 0 \text{ and } W(1) \le 0 \text{ and } W(2) \le 0).$$

42. Let W be Brownian motion. Find the distribution of

$$\text{(a) } \sum_{n=1}^{4} W(n); \quad \text{(b) } \sum_{n=1}^{4} W(1/n); \quad \text{(c) } \int_0^1 W(t)\mathrm{d}t.$$

43. Let W be Brownian motion. Find $\mathbb{P}(\int_0^1 W(t)\mathrm{d}t > 2/\sqrt{3})$.

44. Show that the normal kurtosis is three, i.e., the fourth moment of a normal random variable with zero mean is three times its variance squared.

45. Let $W(t)$, $t \ge 0$, be Brownian motion. Define the *Brownian bridge* by

$$X(t) = W(t) - tW(1), \quad 0 \le t \le 1.$$

Calculate

$$\text{(a) } \mathbb{E}(X(t)); \quad \text{(b) } \mathbb{C}\mathrm{ov}(X(s), X(t)).$$

46. Calculate the third variation of Brownian motion.

47. Let S be geometric Brownian motion with parameters α and σ and let $K > 0$. Show that, for $T > 0$,

$$\mathbb{E}(e^{-\alpha T}(S(T) - K)^+) = S(0)N(d_+(T, S(0))) - Ke^{-\alpha T}N(d_-(T, S(0))),$$

where N is the cumulative standard normal distribution function and

$$d_\pm(T, S(0)) = \frac{1}{\sigma\sqrt{T}}\left\{\log\frac{S(0)}{K} + \left(\alpha \pm \frac{\sigma^2}{2}\right)T\right\}.$$