40. Problems from the Textbook: 1, 3, 5, 7, 9, 11, 15, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 49, 51, 54, 57, 63, 69, 71, 75, 83 (4.4); 2, 3, 5, 11, 13, 19, 20, 21, 26, 27, 40, 43, 46, 49, 51, 55, 64 (4.5); 1, 2, 12, 19, 32 (4.PP); 2, 3, 7, 12, 13, 15, 23, 29, 33, 36, 37, 39, 51 (5.1); 1, 3, 5, 7, 9, 11, 15, 19, 21, 29, 35, 43, 45, 47, 49, 54, 56, 57 (5.2).

41. Find the area under f over the interval I:

    (a) $f(x) = x^2 + x + 2$, $I = [0, 1]$;

    (b) $f(x) = x^2$, $I = [3, 6]$.

42. Let $f(x) = \sqrt{1 - x^2}$, $I = [0, 1]$. Sketch the graph of f. Recall from your trigonometry class what the area under f over I is. Represent this area by the limit of sums A_n (using the definition of the area). Calculate A_1, A_2, A_3, A_4, A_5, A_{10} (and, if you have a computer or enough time, A_n for $n \in \{20, 30, 50, 100, 200, 300, 500, 1000, 5000\}$).

43. (Project – Five extra points; Due Apr 27) Let $L : (0, \infty) \to \mathbb{R}$ be defined by $L(x) = \int_1^x \frac{1}{t} dt$.

    (a) Find $L(1)$.

    (b) Find the intervals where L is positive and negative, respectively. Also find the zeros of L.

    (c) Find $L'(x)$ for $x > 0$.

    (d) Find the intervals where L is increasing and decreasing, respectively. Also find the local extrema of L.

    (e) Find the intervals where L is concave upwards and downwards, respectively. Also find the inflection points of L. Draw a rough sketch of L.

    (f) Show that $L(pq) = L(p) + L(q)$ for $p, q > 0$. (Hint: Consider the function $F(x) = L(px)$ and find its derivative.)

    (g) Show that $L\left(\frac{1}{q}\right) = -L(q)$ for $q > 0$.

    (h) Show that $L\left(\frac{p}{q}\right) = L(p) - L(q)$ for $p, q > 0$.

    (i) Show that $L(p^\alpha) = \alpha L(p)$ for $p > 0$ and $\alpha \in \mathbb{R}$. (Hint: Consider the function $G(x) = L(x^\alpha)$ and find its derivative.)

    (j) Draw $f(t) = \frac{1}{t}$ for $1 \le t \le 3$ as well as the two rectangles with side $[1, 2]$, height $f(2)$ and side $[2, 3]$, height $f(3)$, and calculate the sum of the area of these two rectangles. Hence find a lower bound for $L(3)$. Then, draw the two rectangles with side $[1, 2]$, height $f(1)$ and side $[2, 3]$, height $f(2)$ and hence find an upper bound for $L(3)$.

    (k) This time split the interval $[1, 3]$ in four equally long parts and find, similarly as in (j), a lower and an upper bound (these bounds will be better than the ones in (j)) for $L(3)$.

    (l) Finally, split the interval $[1, 3]$ into eight equally long parts and give the resulting lower and upper bounds. Write a computer program that can do this game with e.g. 1024 tiny intervals.

44. Let $f(t) = \frac{1}{1+t^2}$ and $A(x) = \int_0^x f(t) dt$.

    (a) Find the zeros, local extrema, and inflection points of f and draw the graph of f.

    (b) Give the zeros of A as well as the intervals where A is positive and negative, respectively.

    (c) Show $A(-x) = -A(x)$ and find $A'(x)$.

    (d) Let $p \in \mathbb{R}$ and put $B(x) = A\left(\frac{p-x}{1+px}\right)$. Find and simplify $B'(x)$.

    (e) Show that $A(p) - A(q) = A\left(\frac{p-q}{1+pq}\right)$ holds for all $p, q \in \mathbb{R}$ with $pq > -1$.