Research Article

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Abel dynamic equations of first and second kind

Abstract: This paper gives the definition and analysis of Abel dynamic equations on a general time scale. As such, the results contain as special cases results for classical Abel differential equations and results for new Abel difference equations. By using appropriate transformations, expressions of Abel dynamic equations of second kind are derived on the general time scale. This also leads to a specific class of Abel dynamic equations of first kind. Finally, the canonical Abel dynamic equation is defined and examined.

Keywords: Dynamic equation, time scale, Abel equation

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1 Introduction

Abel equations constitute a generalization of Riccati equations and belong to the class of inhomogeneous nonlinear differential equations of first order. Only some special cases of Abel differential equations of first and second kind are solvable so far and one is still interested in finding methods to obtain solutions to more general classes since Abel equations are used to model real-life problems [6]. To better simulate real-life situations, in this paper we initiate the study of Abel dynamic equations on a general time scale. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers [2, p. 1], and the general calculus for dynamic equations on time scales is construction in such a way that it reduces to the usual calculus for differential equations if $\mathbb{T} = \mathbb{R}$ and the calculus for difference equations if $\mathbb{T} = \mathbb{Z}$. In the following, we recall some basic concepts from the theory of time scales calculus and refer to [2, 3] for details.

Definition 1.1 (see [2, Definition 1.1]). Let $\mathbb{T}$ be a time scale. Then we define

- the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by
$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{for all } t \in \mathbb{T};$$
- the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ by
$$\mu(t) := \sigma(t) - t \quad \text{for all } t \in \mathbb{T}.$$

The derivative of a function $f : \mathbb{T} \to \mathbb{R}$ is denoted by f^Δ and defined in such a way that $f^\Delta(t) = f'(t)$ if $\mathbb{T} = \mathbb{R}$ and $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$ if $\mathbb{T} = \mathbb{Z}$ (see [2] for details). By denoting $f^\sigma = f \circ \sigma$, we then have (subject to the appropriate assumptions on f and g) the formulas

$$f^\sigma = f + \mu f^\Delta, \quad (fg)^\Delta = fg^\Delta + g^\sigma f^\Delta, \quad \left(\frac{f}{g}\right)^\Delta = \frac{fg^\Delta - gf^\Delta}{gg^\sigma}.$$

Definition 1.2 (see [2, Definitions 1.58, 2.25]). A function $p : \mathbb{T} \to \mathbb{R}$ is called

- *regressive* if
$$1 + \mu(t)p(t) \neq 0 \quad \text{for all } t \in \mathbb{T};$$
- *rd-continuous* if p is continuous in all right-dense points and the left-sided limits exist in all left-dense points.

The set of all rd-continuous functions is denoted by C_{rd}, and the set of all regressive and rd-continuous functions is denoted by $\mathcal{R}$.

It is known that, if $p \in \mathcal{R}$, then the initial value problem given in the next definition has a unique solution.

Definition 1.3 (see [2, Section 2.2]). Let $t_0 \in \mathbb{R}$ and $p \in \mathcal{R}$. The unique solution of the initial value problem

$$y^\Delta = p(t)y, \quad y(t_0) = 1$$

is called the *time scales exponential function* and denoted by $e_p(\cdot, t_0)$.

It is also noted that for $p \in \mathcal{R}$, the solution of the initial value problem

$$y^\Delta = -p(t)y^\sigma, \quad y(t_0) = 1$$

is $e_{\ominus p}(\cdot, t_0)$, where

$$\ominus p = -\frac{p}{1 + \mu p}.$$

2 Closed-form solutions of Abel dynamic equations of second kind

Definition 2.1. Let $f_0, f_1, f_2, g_0, g_1 : \mathbb{T} \to \mathbb{R}$. We define the *Abel dynamic equation of second kind* as an equation that has any of the following six forms:

$$\big(g_0(x) + g_1(x)\tilde{u}\big)u^\Delta = f_0(x) + f_1(x)u + f_2(x)u^2, \tag{2.1}$$
$$\big(g_0(x) + g_1(x)\tilde{u}\big)u^\Delta = f_0(x) + f_1(x)u^\sigma + f_2(x)(u^\sigma)^2, \tag{2.2}$$
$$\big(g_0(x) + g_1(x)\tilde{u}\big)u^\Delta = f_0(x) + f_1(x)u^\sigma + f_2(x)u^2, \tag{2.3}$$
$$\big(g_0(x) + g_1(x)\tilde{u}\big)u^\Delta = f_0(x) + f_1(x)u + f_2(x)(u^\sigma)^2, \tag{2.4}$$
$$\big(g_0(x) + g_1(x)\tilde{u}\big)u^\Delta = f_0(x) + f_1(x)\tilde{u} + f_2(x)u^2, \tag{2.5}$$
$$\big(g_0(x) + g_1(x)\tilde{u}\big)u^\Delta = f_0(x) + f_1(x)\tilde{u} + f_2(x)(u^\sigma)^2, \tag{2.6}$$

where $u = u(x)$ and $\tilde{u} = \frac{u+u^\sigma}{2}$.

Example 2.2. If $\mathbb{T} = \mathbb{R}$, then all expressions (2.1)–(2.6) are equivalent to the known Abel differential equation of second kind

$$\big(g_0(x) + g_1(x)u\big)u' = f_0(x) + f_1(x)u + f_2(x)u^2, \quad u = u(x). \tag{2.7}$$

Example 2.3. If $\mathbb{T} = \mathbb{Z}$, the Abel equation of second kind (2.1) becomes

$$\left(g_0 + g_1\frac{u(x+1) + u(x)}{2}\right)\big(u(x+1) - u(x)\big) = f_0 + f_1u(x) + f_2u^2(x),$$

which can be written in the form

$$u(x+1)\big(G_0 + G_1u(x+1)\big) = F_0 + F_1u(x) + F_2u^2(x) \tag{2.8}$$

with

$$G_0 = g_0, \quad G_1 = \frac{g_1}{2}, \quad F_0 = f_0, \quad F_1 = f_1 + g_0, \quad F_2 = f_2 + \frac{g_1}{2}.$$

All other expressions of the Abel dynamic equation of second kind (2.2)–(2.6) have the same form as (2.8). As in the real case (see Example 2.2), in the discrete case there exists only one form of the Abel equation of second kind.

Based on a paper by Bougoffa [4] which presents a solution method for the Abel differential equation of second kind (2.7), one can derive a solution strategy for all expressions of the Abel dynamic equations of second kind (2.1)–(2.6). This is done in the remainder of this section.

Theorem 2.4. *Let $f_0, f_1, f_2 : \mathbb{T} \to \mathbb{R}$ and $g_0, g_1 : \mathbb{T} \to \mathbb{R} \setminus \{0\}$ be such that*

$$\frac{f_0}{g_1} \in \mathrm{C_{rd}} \quad \text{and} \quad p_1, p_2 \in \mathcal{R}, \quad \text{where} \quad p_1 := \frac{f_1}{g_0}, \quad p_2 := \frac{2f_2}{g_1}.$$

Let $x_0 \in \mathbb{T}$ and define

$$B_1(x) := e_{\ominus p_1}(x, x_0) \quad \text{and} \quad B_2(x) := e_{\ominus p_2}(x, x_0),$$

and suppose that

$$\lambda := \frac{2B_2^\sigma(x)g_0(x)}{B_1^\sigma(x)g_1(x)} \quad \text{is independent of } x \in \mathbb{T}.$$

Then the solution of (2.1) *satisfying the initial condition $u(x_0) = u_0 \in \mathbb{R}$ is implicitly given by*

$$B_2(x)u^2(x) + \lambda B_1(x)u(x) = u_0^2 + \lambda u_0 + 2\int_{x_0}^{x} B_2^\sigma(t)\frac{f_0(t)}{g_1(t)}\Delta t.$$

Proof. The given assumptions imply that

$$B_1^\Delta = -\frac{f_1}{g_0}B_1^\sigma, \quad B_2^\Delta = -2\frac{f_2}{g_1}B_2^\sigma$$

and the so-called λ-condition

$$2B_2^\sigma g_0 = \lambda B_1^\sigma g_1$$

holds. These three equations together with a double application of the product rule yield

$$\begin{aligned}(\lambda B_1 u + B_2 u^2)^\Delta &= \lambda(B_1^\sigma u^\Delta + B_1^\Delta u) + 2B_2^\sigma \tilde{u} u^\Delta + B_2^\Delta u^2 \\ &= \lambda B_1^\sigma g_1\left(u^\Delta + \frac{B_1^\Delta}{B_1^\sigma}u\right)\cdot\frac{1}{g_1} + B_2^\sigma\left(2\tilde{u}u^\Delta + \frac{B_2^\Delta}{B_2^\sigma}u^2\right) \\ &= 2B_2^\sigma g_0\left(u^\Delta - \frac{f_1}{g_0}u\right)\cdot\frac{1}{g_1} + B_2^\sigma\left(2\tilde{u}u^\Delta - 2\frac{f_2}{g_1}u^2\right) \\ &= \frac{2B_2^\sigma}{g_1}((g_0 + g_1\tilde{u})u^\Delta - (f_0 + f_1 u + f_2 u^2)) + 2B_2^\sigma\frac{f_0}{g_1},\end{aligned}$$

and the first term after the last equal sign vanishes if and only if u solves (2.1). The integration from x_0 to x and noticing that $B_1(x_0) = B_2(x_0) = 1$ complete the proof. □

To derive an implicit solution for the expressions of the Abel dynamic equations of second kind (2.2)–(2.4), the previous idea is used, although the λ-condition and the solution change. We only state the result for equation (2.2).

Theorem 2.5. *Let $f_0, f_1, f_2 : \mathbb{T} \to \mathbb{R}$ and $g_0, g_1 : \mathbb{T} \to \mathbb{R} \setminus \{0\}$ be such that*

$$\frac{f_0}{g_1} \in \mathrm{C_{rd}} \quad \text{and} \quad -p_1, -p_2 \in \mathcal{R}, \quad \text{where} \quad p_1 := \frac{f_1}{g_0}, \quad p_2 := \frac{2f_2}{g_1}.$$

Let $x_0 \in \mathbb{T}$ and define

$$C_1(x) := e_{-p_1}(x, x_0) \quad \text{and} \quad C_2(x) := e_{-p_2}(x, x_0),$$

and suppose that

$$\lambda := \frac{2C_2(x)g_0(x)}{C_1(x)g_1(x)} \quad \text{is independent of } x \in \mathbb{T}.$$

Then the solution of (2.2) *satisfying the initial condition $u(x_0) = u_0 \in \mathbb{R}$ is implicitly given by*

$$C_2(x)u^2(x) + \lambda C_1(x)u(x) = u_0^2 + \lambda u_0 + 2\int_{x_0}^{x} C_2(t)\frac{f_0(t)}{g_1(t)}\Delta t.$$

Proof. To verify this, one follows the same steps as in the proof of Theorem 2.4. □

For expressions (2.5) and (2.6), an additional condition has to be satisfied in order to get an implicit solution.

Theorem 2.6. *Let $f_0, f_1, f_2 : \mathbb{T} \to \mathbb{R}$ and $g_0, g_1 : \mathbb{T} \to \mathbb{R} \setminus \{0\}$ be such that*

$$\frac{f_0}{g_1} \in \mathrm{C_{rd}} \quad \text{and} \quad p_1, p_2, -p_1 \in \mathcal{R}, \quad \text{where} \quad p_1 := \frac{f_1}{g_0}, \quad p_2 := \frac{2f_2}{g_1}.$$

Let $x_0 \in \mathbb{T}$ and define

$$B_1(x) := e_{\ominus p_1}(x, x_0), \quad B_2(x) := e_{\ominus p_2}(x, x_0), \quad C_1(x) := e_{-p_1}(x, x_0),$$

and suppose that

$$\lambda := \frac{B_2^\sigma(x) g_0(x)}{B_1^\sigma(x) g_1(x)} \quad \text{is independent of } x \in \mathbb{T}$$

and

$$\Lambda := \frac{B_2^\sigma(x) g_0(x)}{C_1(x) g_1(x)} \quad \text{is independent of } x \in \mathbb{T}.$$

Then the solution of (2.5) *satisfying the initial condition $u(x_0) = u_0 \in \mathbb{R}$ is implicitly given by*

$$B_2(x)u^2(x) + \lambda B_1(x)u(x) + \Lambda C_1(x)u(x) = u_0^2 + (\lambda + \Lambda)u_0 + 2\int_{x_0}^{x} B_2^\sigma(t)\frac{f_0(t)}{g_1(t)}\Delta t.$$

Proof. The given assumptions imply that

$$B_1^\Delta = -\frac{f_1}{g_0}B_1^\sigma, \quad B_2^\Delta = -2\frac{f_2}{g_1}B_2^\sigma, \quad C_1^\Delta = -\frac{f_1}{g_0}C_1$$

and the two λ-conditions

$$B_2^\sigma g_0 = \lambda B_1^\sigma g_1 \quad \text{and} \quad B_2^\sigma g_0 = \Lambda C_1 g_1$$

hold. These five equations together with a triple application of the product rule yield

$$\begin{aligned}
(\lambda B_1 u + \Lambda C_1 u + B_2 u^2)^\Delta &= \lambda(B_1 u)^\Delta + \Lambda(C_1 u)^\Delta + (B_2 u^2)^\Delta \\
&= \lambda(B_1^\sigma u^\Delta + B_1^\Delta u) + \Lambda(C_1 u^\Delta + C_1^\Delta u^\sigma) + 2B_2^\sigma \tilde{u} u^\Delta + B_2^\Delta u^2 \\
&= \lambda B_1^\sigma g_1\Big(u^\Delta + \frac{B_1^\Delta}{B_1^\sigma}u\Big)\cdot\frac{1}{g_1} + \Lambda C_1 g_1\Big(u^\Delta + \frac{C_1^\Delta}{C_1}u^\sigma\Big)\cdot\frac{1}{g_1} + B_2^\sigma\Big(2\tilde{u}u^\Delta + \frac{B_2^\Delta}{B_2^\sigma}u^2\Big) \\
&= B_2^\sigma g_0\Big(u^\Delta - \frac{f_1}{g_0}u\Big)\cdot\frac{1}{g_1} + B_2^\sigma g_0\Big(u^\Delta - \frac{f_1}{g_0}u^\sigma\Big)\cdot\frac{1}{g_1} + B_2^\sigma\Big(2\tilde{u}u^\Delta - 2\frac{f_2}{g_1}u^2\Big) \\
&= \frac{2B_2^\sigma}{g_1}((g_0 + g_1\tilde{u})u^\Delta - (f_0 + f_1\tilde{u} + f_2 u^2)) + 2B_2^\sigma\frac{f_0}{g_1},
\end{aligned}$$

and the first term after the last equal sign vanishes if and only if u solves (2.5). The integration from x_0 to x and noticing that $B_1(x_0) = B_2(x_0) = C_1(x_0) = 1$ complete the proof. □

Note that $g_0 \neq 0$ is critical in the above proofs. For an Abel dynamic equation of second kind with $g_0 = 0$, one may use $u = v + \frac{1}{2}$ to obtain an Abel dynamic equation in v that satisfies $g_0 \neq 0$.

3 Special class of Abel dynamic equations of first kind

One can easily prove the following: If u solves an Abel differential equation of second kind

$$(g(x) + u)u' = f_0(x) + f_1(x)u + f_2(x)u^2, \quad u = u(x) \tag{3.1}$$

such that $g(x) + u(x) \neq 0$, then $y = 1/(g + u)$ solves the special class of the Abel differential equations of first kind

$$y' = h_0(x) + h_1(x)y + h_2(x)y^2 + h_3(x)y^3, \quad y = y(x), \tag{3.2}$$

where

$$h_0 = 0, \quad h_1 = -f_2, \quad h_2 = 2f_2 g - f_1 - g', \quad h_3 = f_1 g - f_0 - f_2 g^2. \tag{3.3}$$

We now present the time scales analogue of this transformation in order to obtain the special class of the Abel dynamic equations of first kind.

Definition 3.1. Let $h_1, h_{21}, h_{22}, h_3 : \mathbb{T} \to \mathbb{R}$. We define the special class of the *Abel dynamic equations of first kind* as an equation that has one of the following two forms:

$$\tilde{y} y^\Delta = h_1(x) y^2 + h_{21}(x) y^2 y^\sigma + h_{22}(x) y (y^\sigma)^2 + h_3(x) y^2 (y^\sigma)^2, \tag{3.4}$$

$$\tilde{y} y^\Delta = h_1(x) (y^\sigma)^2 + h_{21}(x) y^2 y^\sigma + h_{22}(x) y (y^\sigma)^2 + h_3(x) y^2 (y^\sigma)^2, \tag{3.5}$$

where $y = y(x)$ and $\tilde{y} = \frac{y + y^\sigma}{2}$.

Theorem 3.2. *Let $f_0, f_1, f_2, g : \mathbb{T} \to \mathbb{R}$. If u solves the Abel dynamic equation of second kind*

$$(\tilde{g}(x) + \tilde{u}) u^\Delta = f_0(x) + f_1(x) u + f_2(x) u^2, \quad u = u(x), \tag{3.6}$$

such that $g(x) + u(x) \neq 0$, then $y = 1/(g + u)$ solves the special class of the Abel dynamic equations of first kind (3.5), where

$$h_1 = -f_2, \quad h_{21} = -\frac{g^\Delta}{2}, \quad h_{22} = 2f_2 g - f_1 - \frac{g^\Delta}{2}, \quad h_3 = f_1 g - f_0 - f_2 g^2.$$

Proof. Suppose u solves (3.6) such that $g(x) + u(x) \neq 0$. Define $y = 1/(g + u)$. Then an application of the quotient rule yields

$$\begin{aligned}
\tilde{y} y^\Delta &= -\frac{\frac{1}{u+g} + \frac{1}{u^\sigma + g^\sigma}}{2} \cdot \frac{u^\Delta + g^\Delta}{(u + g)(u^\sigma + g^\sigma)} \\
&= -(\tilde{g} + \tilde{u})(u^\Delta + g^\Delta) y^2 (y^\sigma)^2 \\
&= -(f_0 + f_1 u + f_2 u^2 + (\tilde{g} + \tilde{u}) g^\Delta) y^2 (y^\sigma)^2 \\
&= -\Big(f_0 + \frac{f_1}{y} - f_1 g + \frac{f_2}{y^2} - \frac{2 f_2 g}{y} + f_2 g^2 + \frac{g^\Delta}{2y} + \frac{g^\Delta}{2 y^\sigma} \Big) y^2 (y^\sigma)^2 \\
&= h_1 (y^\sigma)^2 + h_{21} y^2 y^\sigma + h_{22} y (y^\sigma)^2 + h_3 y^2 (y^\sigma)^2,
\end{aligned}$$

where h_1, h_{21}, h_{22}, h_3 are as given in the statement. Thus y solves (3.5). □

Example 3.3. Note that for $\mathbb{T} = \mathbb{R}$, the coefficients h_1, $h_2 = h_{21} + h_{22}$, and h_3 match the coefficients (3.3). Furthermore, for $\mathbb{T} = \mathbb{R}$, the Abel dynamic equation (3.5) is of the same form as (3.2).

Example 3.4. The Abel differential equation is a generalization of the Bernoulli equation. If the Abel differential equation of first kind satisfies $h_0 = h_2 = 0$, then the Abel equation is

$$y' = h_1 y + h_3 y^3,$$

which is a Bernoulli differential equation with Bernoulli factor $\alpha = 3$. In the case of a general time scale, $h_0 = h_{21} = h_{22} = 0$ yields

$$\tilde{y} y^\Delta = h_1 (y^\sigma)^2 + h_3 (y^\sigma)^2 y^2,$$

which is the already known Bernoulli dynamic equation [1] with Bernoulli factor $\alpha = 3$.

4 Canonical Abel dynamic equation

One can easily prove the following: If u solves the Abel differential equation of second kind (3.1), then

$$w = (u + g) \exp\Big(- \int f_2(x) \mathrm{d}x \Big)$$

solves the canonical form of the Abel differential equation

$$ww' = G_0 + G_1 w, \quad w = w(x),$$

where

$$G_0 = (f_0 - f_1 g + f_2 g^2)\exp\Big(-2\int f_2(x)\mathrm{d}x\Big)$$

and

$$G_1 = (g' + f_1 - 2f_2 g)\exp\Big(-\int f_2(x)\mathrm{d}x\Big)$$

(see, e.g., [5, p. 27]). We now present the time scales analogue of the canonical form and of this transformation.

Definition 4.1. Let $G_0, G_{11}, G_{12} : \mathbb{T} \to \mathbb{R}$. We define the *canonical form of an Abel dynamic equation* as an equation of the following form:

$$\tilde{w}w^\Delta = G_0(x) + G_{11}(x)w + G_{12}(x)w^\sigma, \tag{4.1}$$

where $w = w(x)$ and $\tilde{w} = \frac{w+w^\sigma}{2}$.

Theorem 4.2. *Let $f_0, f_1, f_2, g : \mathbb{T} \to \mathbb{R}$ and $x_0 \in \mathbb{T}$. Assume there exists $f \in \mathcal{R}$ such that*

$$f\Big(1 + \frac{\mu f}{2}\Big) = -f_2(1 + \mu f)^2. \tag{4.2}$$

If u solves the Abel dynamic equation of second kind (3.6), *then $w = (g + u)e_f(\cdot, x_0)$ solves the canonical Abel dynamic equation* (4.1), *where*

$$G_0 = (f_0 - f_1 g + f_2 g^2)(e_f^\sigma(\cdot, x_0))^2, \quad G_{11} = \Big(f_1 - 2f_2 g + \frac{g^\Delta}{2}\Big)(1 + \mu f)e_f^\sigma(\cdot, x_0), \quad G_{12} = \frac{g^\Delta}{2}e_f^\sigma(\cdot, x_0).$$

Proof. Suppose u solves (3.6). Define $w = (g + u)e_f$, where we write short e_f for $e_f(\cdot, x_0)$. Then (4.2) together with the known formulas

$$e_f^\Delta = fe_f, \quad e_f = e_f^\sigma - \mu f e_f, \quad e_f^\sigma = (1 + \mu f)e_f, \quad \mu f^\Delta = f^\sigma - f$$

and the application of the product rule yield

$$\begin{aligned}
\tilde{w}w^\Delta &= \frac{(u+g)e_f + (u^\sigma + g^\sigma)e_f^\sigma}{2}[(u^\Delta + g^\Delta)e_f^\sigma + (u+g)fe_f] \\
&= \Big[(\tilde{u} + \tilde{g})e_f^\sigma - \frac{\mu f}{2}(u+g)e_f\Big][(u^\Delta + g^\Delta)e_f^\sigma + (u+g)fe_f] \\
&= (f_0 + f_1 u + f_2 u^2)(e_f^\sigma)^2 + (\tilde{u} + \tilde{g})g^\Delta(e_f^\sigma)^2 \\
&\qquad - \frac{f}{2}(u+g)(u^\sigma - u + g^\sigma - g)e_f e_f^\sigma + (\tilde{u} + \tilde{g})(u+g)fe_f e_f^\sigma - \frac{\mu f^2}{2}(u+g)^2 e_f^2 \\
&= [f_0 + f_1(u+g) - f_1 g + f_2(u+g)^2 - f_2(2(u+g)g - g^2)](e_f^\sigma)^2 \\
&\qquad + (\tilde{u} + \tilde{g})g^\Delta(e_f^\sigma)^2 + f(u+g)^2 e_f^2\Big[(1 + \mu f) - \frac{\mu f}{2}\Big] \\
&= [f_0 - f_1 g + f_2 g^2](e_f^\sigma)^2 + \Big[f_1 - 2f_2 g + \frac{g^\Delta}{2}\Big](u+g)(e_f^\sigma)^2 + (u^\sigma + g^\sigma)\frac{g^\Delta}{2}(e_f^\sigma)^2 \\
&= G_0 + G_{11}w + G_{12}w^\sigma,
\end{aligned}$$

where G_0, G_{11}, G_{12} are as given in the statement. Thus w solves (4.1). □

Example 4.3. Note that for $\mathbb{T} = \mathbb{R}$, (4.2) yields $f = -f_2$, and therefore the transformation is the same one as given at the beginning of this section. Also note how G_0 in Theorem 4.2 matches G_0 given at the beginning of this section and how the sum of G_{11} and G_{12} from Theorem 4.2 matches G_1 given at the beginning of this section.

The same canonical form can also be achieved by applying a transformation to the Abel equation of first kind. To be more precise, one can easily prove the following: If y solves the Abel differential equation of first kind (3.2) such that $y(t) \neq 0$, then

$$w = \frac{\exp(\int h_1(x)\mathrm{d}x)}{y}$$

solves the canonical form of the Abel differential equation

$$ww' = G_0 + G_1 w, \quad w = w(x),$$

where

$$G_0 = -h_3 \exp\Big(2\int h_1(x)\mathrm{d}x\Big) \quad \text{and} \quad G_1 = -h_2 \exp\Big(\int h_1(x)\mathrm{d}x\Big).$$

We now present the time scales analogue of this transformation.

Theorem 4.4. *Let $h_1, h_{21}, h_{22}, h_3 : \mathbb{T} \to \mathbb{R}$ and $x_0 \in \mathbb{T}$. Assume there exists $h \in \mathcal{R}$ such that*

$$h\Big(1 + \frac{\mu h}{2}\Big) = h_1. \tag{4.3}$$

If y solves the Abel dynamic equation of first kind (3.4) *such that $y(x) \neq 0$, then $w = e_h(\cdot, x_0)/y$ solves the canonical Abel dynamic equation* (4.1), *where*

$$G_0 = -h_3(e_h(\cdot, x_0))^2, \quad G_{11} = -h_{22}e_h(\cdot, x_0), \quad G_{12} = -\frac{h_{21}e_h(\cdot, x_0)}{1 + \mu h}.$$

Proof. Suppose y solves (3.4) such that $y(x) \neq 0$. Define $w = e_h/y$, where we write short e_h for $e_h(\cdot, x_0)$. Then (4.3) together with the known formulas

$$e_h^\Delta = he_h, \quad e_h = e_h^\sigma - \mu he_h, \quad e_h^\sigma = (1 + \mu h)e_h, \quad \mu h^\Delta = h^\sigma - h$$

and the application of the quotient rule yield

$$\begin{aligned}
\tilde{w}w^\Delta &= \frac{\frac{e_h}{y} + \frac{e_h^\sigma}{y^\sigma}}{2} \frac{he_h y - e_h y^\Delta}{yy^\sigma} \\
&= \frac{he_h^2}{2yy^\sigma} + \frac{he_h e_h^\sigma}{2(y^\sigma)^2} - \frac{e_h y^\sigma + (e_h + \mu he_h)y}{2yy^\sigma} e_h \frac{y^\Delta}{yy^\sigma} \\
&= \frac{he_h^2}{2yy^\sigma} + \frac{he_h e_h^\sigma}{2(y^\sigma)^2} - \Big[\frac{e_h \tilde{y}}{yy^\sigma} + \frac{\mu he_h}{2y^\sigma}\Big] e_h \frac{y^\Delta}{yy^\sigma} \\
&= \frac{he_h^2}{2yy^\sigma} + \frac{he_h e_h^\sigma}{2(y^\sigma)^2} - \frac{e_h^2}{y^2(y^\sigma)^2}\tilde{y}y^\Delta - \frac{he_h^2(y^\sigma - y)}{2y(y^\sigma)^2} \\
&= \frac{he_h e_h^\sigma + he_h^2}{2(y^\sigma)^2} - \frac{e_h^2 h_1}{(y^\sigma)^2} - \frac{e_h^2 h_{21}}{y^\sigma} - \frac{h_{22}e_h^2}{y} - h_3 e_h^2 \\
&= -\frac{e_h h_{21}}{1 + \mu h} w^\sigma - e_h h_{22} w - h_3 e_h^2 \\
&= G_0 + G_{11} w + G_{12} w^\sigma,
\end{aligned}$$

where G_0, G_{11}, G_{12} are as given in the statement. Thus w solves (4.1). □

A similar theorem can also be proved for the other Abel dynamic equation of first kind (3.5). We only state the result for (3.5).

Theorem 4.5. *Let $h_1, h_{21}, h_{22}, h_3 : \mathbb{T} \to \mathbb{R}$ and $x_0 \in \mathbb{T}$. Assume there exists $h \in \mathcal{R}$ such that*

$$h\Big(1 + \frac{\mu h}{2}\Big) = h_1(1 + \mu h)^2. \tag{4.4}$$

If y solves the Abel dynamic equation of first kind (3.5) *such that $y(x) \neq 0$, then $w = e_h(\cdot, x_0)/y$ solves the canonical Abel dynamic equation* (4.1), *where*

$$G_0 = -h_3(e_h^\sigma(\cdot, x_0))^2, \quad G_{11} = -h_{22}e_h^\sigma(\cdot, x_0)(1 + \mu h), \quad G_{12} = -h_{21}e_h^\sigma(\cdot, x_0).$$

Proof. To verify this, one follows the same steps as in the proof of Theorem 4.4. □

Example 4.6. Note that for $\mathbb{T} = \mathbb{R}$, both (4.3) and (4.4) yield $h = h_1$, and therefore the transformations are the same as the one given earlier. Also note how the G_0 in Theorem 4.4 and Theorem 4.5 matches G_0 given earlier and how the sum of G_{11} and G_{12} from Theorem 4.4 and Theorem 4.5 matches G_1 given earlier.

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