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Comparison Theorems for Oscillation of Second-Order Neutral Dynamic Equations

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Abstract. We study oscillation of certain second-order neutral dynamic equations under the assumptions that allow applications to dynamic equations with both delayed and advanced arguments. Some new comparison criteria are presented that can be used in cases where known theorems fail to apply.

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1. Introduction

This work is concerned with oscillatory behavior of a class of second-order neutral dynamic equations

$$(r(t)(x(t)+p(t)x(\eta(t)))^{\Delta})^{\Delta}+q(t)x(g(t))=0 \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$, where r, p, and q are real-valued positive rd-continuous functions on $\mathbb{T}$, η, $g:\mathbb{T}\to\mathbb{T}$ are rd-continuous, and $\lim_{t\to\infty}\eta(t)=\lim_{t\to\infty}g(t)=\infty$. The increasing interest in oscillatory properties of solutions to second-order dynamic equations is motivated by their applications in the natural sciences and engineering. We refer the reader to [2,4,5,8–16,18–36] and the references cited therein.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above. We assume $t_0\in\mathbb{T}$ and it is convenient to assume $t_0>0$, and define the time scale interval of the form $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$. On any time scale we define the forward and backward jump operators by

$$\sigma(t):=\inf\{s\in\mathbb{T}|s>t\}\quad\text{and}\quad\rho(t):=\sup\{s\in\mathbb{T}|s<t\},$$

where $\inf\emptyset:=\sup\mathbb{T}$ and $\sup\emptyset:=\inf\mathbb{T}$, and $\emptyset$ denotes the empty set. A point $t\in\mathbb{T}$ is said to be left-dense if $\rho(t)=t$ and $t>\inf\mathbb{T}$, right-dense if $\sigma(t)=t$

and $t < \sup\mathbb{T}$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess μ of the time scale is defined by $\mu(t) := \sigma(t) - t$. For other details on time scales, we refer the reader to [1,6,7,17].

By a solution of (1.1) we mean a nontrivial real-valued function $x \in \mathrm{C}^1_{\mathrm{rd}}[T_x, \infty)_{\mathbb{T}}$, $T_x \in [t_0, \infty)_{\mathbb{T}}$ which has the properties that $x + p \cdot x \circ \eta$ and $r(x + p \cdot x \circ \eta)^{\Delta}$ are defined and Δ-differentiable for $t \in \mathbb{T}$ and satisfies (1.1) on $[T_x, \infty)_{\mathbb{T}}$. The solutions vanishing in some neighbourhood of infinity will be excluded from our consideration. As usual, a solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

In what follows, we briefly comment on related results that motivated our study. Agarwal et al. [4], Chen [8], Erbe et al. [10], Han et al. [15], Li et al. [19], Şahiner [23], Saker [25], Saker et al. [28], Saker and O'Regan [29], Wu et al. [33], Zhang et al. [34], and Zhang and Wang [36] obtained some oscillation criteria for (1.1) in the case where

$$\int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \infty.$$

However, there are very few results for oscillation of (1.1) under the assumption

$$\int_{t_0}^{\infty} \frac{\Delta t}{r(t)} < \infty. \tag{1.2}$$

Thereinto, Saker [26] established some oscillation theorems for (1.1) provided that

$$0 \le p(t) < 1, \quad p^{\Delta}(t) \ge 0, \quad g(t) \le \eta(t) \le t, \quad \eta^{\Delta}(t) \ge 0, \tag{1.3}$$

and

$$\int_{T}^{\infty} \frac{1}{r(s)} \int_{T}^{s} q(u)(1 - p(u)) \int_{u}^{\infty} \frac{\Delta t}{r(t)} \Delta u \Delta s = \infty, \tag{1.4}$$

for some $T \ge t_0$. Assuming (1.2), Thandapani et al. [31] obtained some sufficient conditions which guarantee that every solution of (1.1) is oscillatory or tends to zero as $t \to \infty$. Very recently, Li et al. [21] considered (1.1) in the case $\mathbb{T} = \mathbb{R}$,

$$0 \le p(t) \le p_0 < \infty, \quad \text{and} \quad \eta \circ g = g \circ \eta. \tag{1.5}$$

The objective of this paper is to derive several new oscillation criteria for (1.1) under the assumption that (1.2) holds and without requiring restrictive conditions such as (1.3), (1.4), and (1.5). This paper is organized as follows. In Sect. 2, we shall establish some oscillation theorems for (1.1). In Sect. 3, we provide some conclusions to summarize the results obtained.

2. Main Results

All occurring functional inequalities considered in this section are assumed to hold eventually, that is, they are satisfied for all t large enough. In what follows, we use the notation

$$z(t) := x(t) + p(t)x(\eta(t)) \quad \text{and} \quad R(t) := \int_t^\infty \frac{\Delta s}{r(s)}.$$

Theorem 2.1. *Assume* (1.2) *and let*

$$\eta(t) \le t, \quad g(t) \le \sigma(t), \quad 0 \le p(t) < 1, \quad r(t)R(t) - \mu(t) > 0$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$. *Assume further that there exist positive real-valued* Δ*-differentiable functions* v, m *such that*

$$\frac{v(t)}{r(t)\int_{t_1}^t \frac{\Delta s}{r(s)}} - v^\Delta(t) \le 0 \tag{2.1}$$

for all sufficiently large t_1 *and*

$$\frac{m(t)}{r(t)R(t)} + m^\Delta(t) \le 0, \quad 1 - p(t)\frac{m(\eta(t))}{m(t)} > 0. \tag{2.2}$$

If the second-order dynamic equations

$$(ru^\Delta)^\Delta(t) + q(t)(1 - p(g(t)))\frac{v(g(t))}{v^\sigma(t)}u^\sigma(t) = 0 \tag{2.3}$$

and

$$(ru^\Delta)^\Delta(t) + q(t)\left(1 - p(g(t))\frac{m(\eta(g(t)))}{m(g(t))}\right)u^\sigma(t) = 0 \tag{2.4}$$

are oscillatory, then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume $x(t) > 0$, $x(\eta(t)) > 0$, and $x(g(t)) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. In view of (1.1), we obtain

$$(rz^\Delta)^\Delta(t) = -q(t)x(g(t)) < 0 \quad \text{for} \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{2.5}$$

Hence rz^Δ is strictly decreasing, and so there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $z^\Delta(t) > 0$ or $z^\Delta(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. We consider each of the two cases separately.

Case 1 Assume $z^\Delta(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Then we have

$$x(t) = z(t) - p(t)x(\eta(t)) \ge (1 - p(t))z(t). \tag{2.6}$$

Hence by (2.5) and (2.6), we get

$$(rz^\Delta)^\Delta(t) \le -q(t)(1 - p(g(t)))z(g(t)).$$

On the other hand, we obtain

$$z(t) = z(t_1) + \int_{t_1}^t \frac{r(s)z^\Delta(s)}{r(s)}\Delta s \ge \left(r(t)\int_{t_1}^t \frac{\Delta s}{r(s)}\right)z^\Delta(t).$$

Hence,

$$\left(\frac{z}{v}\right)^{\Delta}(t) = \frac{z^{\Delta}(t)v(t) - z(t)v^{\Delta}(t)}{v(t)v^{\sigma}(t)}$$
$$\leq \frac{z(t)}{v(t)v^{\sigma}(t)}\left(\frac{v(t)}{r(t)\int_{t_1}^{t}\frac{\Delta s}{r(s)}} - v^{\Delta}(t)\right) \leq 0,$$

and thus z/v is nonincreasing. We set

$$\omega(t) := \frac{r(t)z^{\Delta}(t)}{z(t)}. \tag{2.7}$$

Then, we get

$$\begin{aligned}\omega^{\Delta}(t) &= \frac{z(t)(rz^{\Delta})^{\Delta}(t) - r(t)(z^{\Delta}(t))^2}{z(t)z^{\sigma}(t)}\\ &\leq -q(t)(1-p(g(t)))\frac{v(g(t))}{v^{\sigma}(t)} - \frac{\omega^2(t)}{r(t)}\frac{z(t)}{z^{\sigma}(t)}\\ &= -q(t)(1-p(g(t)))\frac{v(g(t))}{v^{\sigma}(t)} - \frac{\omega^2(t)}{r(t)}\frac{z(t)}{z(t)+\mu(t)z^{\Delta}(t)}.\end{aligned}$$

Thus,

$$\omega^{\Delta}(t) + q(t)(1-p(g(t)))\frac{v(g(t))}{v^{\sigma}(t)} + \frac{\omega^2(t)}{r(t)+\mu(t)\omega(t)} \leq 0$$

for all large t. Therefore, we get by results of [9] that Eq. (2.3) is nonoscillatory. This is a contradiction.

Case 2 Assume $z^{\Delta}(t) < 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Define the function ω by (2.7). Then $\omega(t) < 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. By (2.5), we get

$$z^{\Delta}(s) \leq \frac{r(t)}{r(s)}z^{\Delta}(t), \quad s \in [t,\infty)_{\mathbb{T}}.$$

Integrating this from t to l, we have

$$z(l) \leq z(t) + r(t)z^{\Delta}(t)\int_t^l \frac{\Delta s}{r(s)}, \quad l \in [t,\infty)_{\mathbb{T}}.$$

Letting $l \to \infty$ in the latter inequality, we get

$$z(t) + r(t)z^{\Delta}(t)R(t) \geq 0, \quad t \in [t_1,\infty)_{\mathbb{T}}.$$

Thus, we obtain

$$\omega(t) \geq -\frac{1}{R(t)}. \tag{2.8}$$

It follows from (2.8) that

$$\frac{z^{\Delta}(t)}{z(t)} \geq -\frac{1}{r(t)R(t)}.$$

Then, we have

$$\begin{aligned}\left(\frac{z}{m}\right)^{\Delta}(t) &= \frac{z^{\Delta}(t)m(t) - z(t)m^{\Delta}(t)}{m(t)m^{\sigma}(t)} \\ &\geq -\frac{z(t)}{m(t)m^{\sigma}(t)}\left(\frac{m(t)}{r(t)R(t)} + m^{\Delta}(t)\right) \geq 0.\end{aligned}$$

Thus, z/m is nondecreasing and so

$$\begin{aligned}x(t) &= z(t) - p(t)x(\eta(t)) \geq z(t) - p(t)z(\eta(t)) \\ &\geq z(t) - p(t)\frac{m(\eta(t))}{m(t)}z(t) = \left(1 - p(t)\frac{m(\eta(t))}{m(t)}\right)z(t).\end{aligned}$$

Differentiating ω and using (2.5), we obtain

$$\begin{aligned}\omega^{\Delta}(t) &\leq -q(t)\left(1 - p(g(t))\frac{m(\eta(g(t)))}{m(g(t))}\right) - \frac{r(t)(z^{\Delta}(t))^2}{z(t)z(\sigma(t))} \\ &= -q(t)\left(1 - p(g(t))\frac{m(\eta(g(t)))}{m(g(t))}\right) - \frac{\omega^2(t)}{r(t)}\frac{z(t)}{z(\sigma(t))} \\ &= -q(t)\left(1 - p(g(t))\frac{m(\eta(g(t)))}{m(g(t))}\right) - \frac{\omega^2(t)}{r(t)}\frac{z(t)}{z(t) + \mu(t)z^{\Delta}(t)} \\ &= -q(t)\left(1 - p(g(t))\frac{m(\eta(g(t)))}{m(g(t))}\right) - \frac{\omega^2(t)}{r(t) + \mu(t)\omega(t)}.\end{aligned}$$

Since

$$r(t) + \mu(t)\omega(t) \geq \frac{r(t)R(t) - \mu(t)}{R(t)} > 0$$

and ω satisfies

$$\omega^{\Delta}(t) + q(t)\left(1 - p(g(t))\frac{m(\eta(g(t)))}{m(g(t))}\right) + \frac{\omega^2(t)}{r(t) + \mu(t)\omega(t)} \leq 0$$

for all large t, we get by results of [9] that Eq. (2.4) is nonoscillatory. This contradiction proves the result. □

With a proof similar to the proof of Theorem 2.1, we can obtain the following result.

Theorem 2.2. *Assume* (1.2) *and let*

$$\eta(t) \leq t, \quad g(t) \geq \sigma(t), \quad 0 \leq p(t) < 1, \quad r(t)R(t) - \mu(t) > 0$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$*. Assume also that there exist positive real-valued* Δ*-differentiable functions* v*,* m *such that* (2.1) *and* (2.2) *hold for all sufficiently large* t_1*. If the second-order dynamic equations*

$$(ru^{\Delta})^{\Delta}(t) + q(t)(1 - p(g(t)))u^{\sigma}(t) = 0 \tag{2.9}$$

and

$$(ru^{\Delta})^{\Delta}(t) + q(t)\left(1 - p(g(t))\frac{m(\eta(g(t)))}{m(g(t))}\right)\frac{m(g(t))}{m^{\sigma}(t)}u^{\sigma}(t) = 0 \tag{2.10}$$

are oscillatory, then (1.1) *is oscillatory.*

Theorem 2.3. *Assume* (1.2) *and let*

$$\eta(t) \geq t, \quad g(t) \leq \sigma(t), \quad 0 \leq p(t) < 1, \quad r(t)R(t) - \mu(t) > 0$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$. *Assume further that there exists a positive real-valued* Δ*-differentiable function* v *such that*

$$\frac{v(t)}{r(t)\int_{t_1}^{t}\frac{\Delta s}{r(s)}} - v^{\Delta}(t) \leq 0, \quad 1 - p(t)\frac{v(\eta(t))}{v(t)} > 0 \tag{2.11}$$

for all sufficiently large t_1. *If the second-order dynamic equations*

$$(ru^{\Delta})^{\Delta}(t) + q(t)\left(1 - p(g(t))\frac{v(\eta(g(t)))}{v(g(t))}\right)\frac{v(g(t))}{v^{\sigma}(t)}u^{\sigma}(t) = 0 \tag{2.12}$$

and

$$(ru^{\Delta})^{\Delta}(t) + q(t)\left(1 - p(g(t))\right)u^{\sigma}(t) = 0 \tag{2.13}$$

are oscillatory, then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume $x(t) > 0$, $x(\eta(t)) > 0$, and $x(g(t)) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. In view of (1.1), we obtain (2.5). Hence, rz^{Δ} is strictly decreasing and so there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $z^{\Delta}(t) > 0$, or $z^{\Delta}(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$.

Case 1 Assume $z^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. With a proof similar to the proof of case 1 in Theorem 2.1, we see that z/v is nonincreasing. Then, we have

$$x(t) = z(t) - p(t)x(\eta(t)) \geq \left(1 - p(t)\frac{v(\eta(t))}{v(t)}\right)z(t).$$

The rest of the proof is similar to that of case 1 in Theorem 2.1.

Case 2 Assume $z^{\Delta}(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. We get

$$x(t) = z(t) - p(t)x(\eta(t)) \geq (1 - p(t))\,z(t).$$

The remainder of the proof is similar to that of case 2 in Theorem 2.1. □

With a proof similar to the proof of Theorems 2.1 and 2.3, we have the following result.

Theorem 2.4. *Assume* (1.2) *and let*

$$\eta(t) \geq t, \quad g(t) \geq \sigma(t), \quad 0 \leq p(t) < 1, \quad r(t)R(t) - \mu(t) > 0$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$. *Assume also that there exist positive real-valued* Δ*-differentiable functions* v, m *such that* (2.11) *holds for all sufficiently large* t_1, *and*

$$\frac{m(t)}{r(t)R(t)} + m^{\Delta}(t) \leq 0. \tag{2.14}$$

If the second-order dynamic equations

$$(ru^{\Delta})^{\Delta}(t) + q(t)\left(1 - p(g(t))\frac{v(\eta(g(t)))}{v(g(t))}\right)u^{\sigma}(t) = 0 \tag{2.15}$$

and

$$(ru^{\Delta})^{\Delta}(t) + q(t)\,(1 - p(g(t)))\,\frac{m(g(t))}{m^{\sigma}(t)}u^{\sigma}(t) = 0 \tag{2.16}$$

are oscillatory, then (1.1) *is oscillatory.*

Now, we shall establish some oscillation results for (1.1) when $p(t) > 1$. We let η^{-1} be the inverse function of η.

Theorem 2.5. *Assume* (1.2), η *is strictly increasing, and let*

$$\eta(t) \geq t, \quad \eta(\sigma(t)) \geq g(t), \quad p(t) > 1, \quad r(t)R(t) - \mu(t) > 0$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$. *Assume further that there exist positive real-valued* Δ*-differentiable functions* v, m *such that* (2.1) *holds for all sufficiently large* t_1, *and*

$$\frac{m(t)}{r(t)R(t)} + m^{\Delta}(t) \leq 0, \quad 1 - \frac{1}{p(\eta^{-1}(t))}\frac{m(\eta^{-1}(t))}{m(t)} > 0. \tag{2.17}$$

If the second-order dynamic equations

$$\begin{aligned} 0 = (ru^{\Delta})^{\Delta}(t) &+ \frac{q(t)}{p(\eta^{-1}(g(t)))} \\ &\times \left(1 - \frac{1}{p(\eta^{-1}(\eta^{-1}(g(t))))}\right)\frac{v(\eta^{-1}(g(t)))}{v^{\sigma}(t)}u^{\sigma}(t) \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} 0 = (ru^{\Delta})^{\Delta}(t) &+ \frac{q(t)}{p(\eta^{-1}(g(t)))} \\ &\times \left(1 - \frac{1}{p(\eta^{-1}(\eta^{-1}(g(t))))}\frac{m(\eta^{-1}(\eta^{-1}(g(t))))}{m(\eta^{-1}(g(t)))}\right)u^{\sigma}(t) \end{aligned} \tag{2.19}$$

are oscillatory, then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume $x(t) > 0$, $x(\eta(t)) > 0$, and $x(g(t)) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. It follows from the definition of z (see also [3, (8.6)]) that

$$\begin{aligned} x(t) &= \frac{1}{p(\eta^{-1}(t))}\left(z(\eta^{-1}(t)) - x(\eta^{-1}(t))\right) \\ &= \frac{z(\eta^{-1}(t))}{p(\eta^{-1}(t))} - \frac{1}{p(\eta^{-1}(t))}\left(\frac{z(\eta^{-1}(\eta^{-1}(t)))}{p(\eta^{-1}(\eta^{-1}(t)))} - \frac{x(\eta^{-1}(\eta^{-1}(t)))}{p(\eta^{-1}(\eta^{-1}(t)))}\right) \\ &\geq \frac{z(\eta^{-1}(t))}{p(\eta^{-1}(t))} - \frac{1}{p(\eta^{-1}(t))}\frac{z(\eta^{-1}(\eta^{-1}(t)))}{p(\eta^{-1}(\eta^{-1}(t)))}. \end{aligned} \tag{2.20}$$

In view of (1.1), we obtain (2.5). Hence rz^{Δ} is strictly decreasing, and so there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $z^{\Delta}(t) > 0$, or $z^{\Delta}(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. We consider each of the two cases separately.

Case 1 Assume $z^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. By virtue of $\eta(t) \geq t$ and (2.20), we see that

$$x(t) \geq \frac{1}{p(\eta^{-1}(t))}\left(1 - \frac{1}{p(\eta^{-1}(\eta^{-1}(t)))}\right)z(\eta^{-1}(t)).$$

Hence by (2.5), we have

$$(rz^{\Delta})^{\Delta}(t) \leq -\frac{q(t)}{p(\eta^{-1}(g(t)))}\left(1-\frac{1}{p(\eta^{-1}(\eta^{-1}(g(t))))}\right)z(\eta^{-1}(g(t))).$$

On the other hand, with a proof similar to the proof of case 1 in Theorem 2.1, we find that z/v is nonincreasing. The remainder of the proof is similar to that of case 1 in Theorem 2.1.

Case 2 Assume $z^{\Delta}(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. From the proof of Theorem 2.1, we obtain that z/m is nondecreasing. It follows from (2.20) that

$$x(t) \geq \frac{1}{p(\eta^{-1}(t))}\left(1-\frac{1}{p(\eta^{-1}(\eta^{-1}(t)))}\frac{m(\eta^{-1}(\eta^{-1}(t)))}{m(\eta^{-1}(t))}\right)z(\eta^{-1}(t)).$$

The rest of the proof is similar to that of case 2 in Theorem 2.1. □

With a proof similar to the proof of Theorems 2.1 and 2.5, we establish the following result.

Theorem 2.6. *Assume* (1.2), *η is strictly increasing, and let*

$$\eta(t) \geq t, \quad \eta(\sigma(t)) \leq g(t), \quad p(t) > 1, \quad r(t)R(t) - \mu(t) > 0$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$. Assume also that there exist positive real-valued Δ-differentiable functions v, m such that (2.1) *and* (2.17) *hold for all sufficiently large t_1. If the second-order dynamic equations*

$$0 = (ru^{\Delta})^{\Delta}(t) + \frac{q(t)}{p(\eta^{-1}(g(t)))}\left(1-\frac{1}{p(\eta^{-1}(\eta^{-1}(g(t))))}\right)u^{\sigma}(t) \quad (2.21)$$

and

$$0 = (ru^{\Delta})^{\Delta}(t) + \frac{q(t)}{p(\eta^{-1}(g(t)))} \times\left(1-\frac{m(\eta^{-1}(\eta^{-1}(g(t))))}{p(\eta^{-1}(\eta^{-1}(g(t))))m(\eta^{-1}(g(t)))}\right)\frac{m(\eta^{-1}(g(t)))}{m^{\sigma}(t)}u^{\sigma}(t) \quad (2.22)$$

are oscillatory, then (1.1) *is oscillatory.*

Theorem 2.7. *Assume* (1.2), *η is strictly increasing, and let*

$$\eta(t) \leq t, \quad \eta(\sigma(t)) \geq g(t), \quad p(t) > 1, \quad r(t)R(t) - \mu(t) > 0$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$. Assume further that there exists a positive real-valued Δ-differentiable function v such that

$$\frac{v(t)}{r(t)\int_{t_1}^{t}\frac{\Delta s}{r(s)}} - v^{\Delta}(t) \leq 0, \quad 1-\frac{1}{p(\eta^{-1}(t))}\frac{v(\eta^{-1}(t))}{v(t)} > 0 \quad (2.23)$$

for all sufficiently large t_1. If the second-order dynamic equations

$$0 = (ru^{\Delta})^{\Delta}(t) + \frac{q(t)}{p(\eta^{-1}(g(t)))} \times\left(1-\frac{v(\eta^{-1}(\eta^{-1}(g(t))))}{p(\eta^{-1}(\eta^{-1}(g(t))))v(\eta^{-1}(g(t)))}\right)\frac{v(\eta^{-1}(g(t)))}{v^{\sigma}(t)}u^{\sigma}(t) \quad (2.24)$$

and

$$0 = (ru^{\Delta})^{\Delta}(t) + \frac{q(t)}{p(\eta^{-1}(g(t)))}\left(1 - \frac{1}{p(\eta^{-1}(\eta^{-1}(g(t))))}\right)u^{\sigma}(t) \tag{2.25}$$

are oscillatory, then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume $x(t) > 0$, $x(\eta(t)) > 0$, and $x(g(t)) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. It follows from the definition of z that (2.20) holds. In view of (1.1), we obtain (2.5). Hence rz^{Δ} is strictly decreasing, and so there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $z^{\Delta}(t) > 0$, or $z^{\Delta}(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$.

Case 1 Assume $z^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. With a proof similar to the proof of case 1 in Theorem 2.1, we see that z/v is nonincreasing. By $\eta(t) \le t$ and (2.20), we have

$$x(t) \ge \frac{1}{p(\eta^{-1}(t))}\left(1 - \frac{v(\eta^{-1}(\eta^{-1}(t)))}{p(\eta^{-1}(\eta^{-1}(t)))v(\eta^{-1}(t))}\right)z(\eta^{-1}(t)). \tag{2.26}$$

The remainder of the proof is similar to that of case 1 in Theorem 2.1.

Case 2 Assume $z^{\Delta}(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. It follows from (2.20) that

$$x(t) \ge \frac{1}{p(\eta^{-1}(t))}\left(1 - \frac{1}{p(\eta^{-1}(\eta^{-1}(t)))}\right)z(\eta^{-1}(t)). \tag{2.27}$$

The rest of the proof is similar to that of case 2 in Theorem 2.1. $\square$

Theorem 2.8. *Assume* (1.2), η *is strictly increasing, and let*

$$\eta(t) \le t, \quad \eta(\sigma(t)) \le g(t), \quad p(t) > 1, \quad r(t)R(t) - \mu(t) > 0$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$. *Assume also that there exist positive real-valued* Δ*-differentiable functions* v, m *such that* (2.14) *and* (2.23) *hold for all sufficiently large* t_1. *If the second-order dynamic equations*

$$\begin{aligned} 0 = (ru^{\Delta})^{\Delta}(t) &+ \frac{q(t)}{p(\eta^{-1}(g(t)))} \\ &\times \left(1 - \frac{v(\eta^{-1}(\eta^{-1}(g(t))))}{p(\eta^{-1}(\eta^{-1}(g(t))))v(\eta^{-1}(g(t)))}\right)u^{\sigma}(t) \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} 0 = (ru^{\Delta})^{\Delta}(t) &+ \frac{q(t)}{p(\eta^{-1}(g(t)))} \\ &\times \left(1 - \frac{1}{p(\eta^{-1}(\eta^{-1}(g(t))))}\right)\frac{m(\eta^{-1}(g(t)))}{m^{\sigma}(t)}u^{\sigma}(t) = 0 \end{aligned} \tag{2.29}$$

are oscillatory, then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume $x(t) > 0$, $x(\eta(t)) > 0$, and $x(g(t)) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. It follows from the definition of z that (2.20) holds. In view of (1.1), we obtain (2.5). Hence rz^{Δ} is strictly decreasing, and so there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $z^{\Delta}(t) > 0$, or $z^{\Delta}(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$.

Case 1 Assume $z^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. With a proof similar to the proof of case 1 in Theorem 2.1, we find that z/v is nonincreasing. From $\eta(t) \le t$ and (2.20), we see that (2.26) holds. The remainder of the proof is similar to that of case 1 in Theorem 2.1.

Case 2 Assume $z^{\Delta}(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. From the proof of Theorem 2.1, we obtain that z/m is nondecreasing. It follows from (2.20) that (2.27) holds. The rest of the proof is similar to that of case 2 in Theorem 2.1. $\square$

Remark 2.9. All our conclusions can be easily extended to a nonlinear second-order neutral dynamic equation

$$\left(r(t)(x(t) + p(t)x(\eta(t)))^{\Delta}\right)^{\Delta} + q(t)f(x(g(t))) = 0. \tag{2.30}$$

Assuming additional condition

$$\frac{f(y)}{y} \ge k > 0 \quad \text{for} \quad y \ne 0 \quad \text{and some constant} \quad k,$$

the reader can verify that the results obtained in this paper hold for (2.30), provided that we replace in the assumptions of our achievements the function q by kq.

3. Conclusions

(i) There are many results on oscillation of equations of the form

$$(au^{\Delta})^{\Delta}(t) + b(t)u^{\sigma}(t) = 0, \tag{3.1}$$

where $t \in [t_0, \infty)_{\mathbb{T}}$, $a(t) > 0$, $b(t) > 0$, and $\int_{t_0}^{\infty} \frac{\Delta t}{a(t)} < \infty$; see, for example, [13,14,16,22,30]. In particular, Hassan [16] used a generalized Riccati transformation to obtain some Kamenev-type oscillation criteria for (3.1), and Řehák [22] established several Hille–Nehari theorems for oscillation of (3.1). Hence, one can obtain some classes of corollaries from Theorem 2.1–Theorem 2.8. The details are left to the reader.

(ii) One can get some criteria for (1.1) by choosing v and m, e.g., let $v(t) = \int_{t_1}^{t} \frac{\Delta s}{r(s)}$ and $m(t) = R(t)$. The details are left to the reader.

(iii) We stress that the study of oscillatory properties of Eq. (1.1) in the case (1.2) brings additional difficulties. In particular, to deal with the case where $z^{\Delta} < 0$ (which is simply eliminated if $\int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \infty$), we have to impose additional assumptions on p. Since the sign of the derivative z^{Δ} is not known, our theorems for oscillation of (1.1) include a pair of assumptions, cf. for example, (2.3) and (2.4). On the other hand, we point out that, contrary to [21,26], we do not need conditions (1.3), (1.4), and (1.5) in our oscillation criteria which, in some sense, is a significant improvement compared to the results in the cited papers. However, this improvement has been achieved at the cost of imposing additional conditions on p.

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