Rendiconti del Circolo Matematico di Palermo Series 2 (2023) 72:467–482
https://doi.org/10.1007/s12215-021-00674-y

Dynamic Local and Nonlocal Initial Value Problems in Banach Spaces

Sanket Tikare[1] · Martin Bohner[2] · Bipan Hazarika[3] · Ravi P. Agarwal[4]

Received: 30 December 2020 / Accepted: 11 September 2021 / Published online: 8 November 2021

Abstract
The objective of this paper is to study the existence of solutions for two classes of dynamic initial value problems in Banach spaces. Our approach is based on the concept of measure of noncompactness and fixed point theorems of Sadovskiĭ and Mönch. We provide some new examples to illustrate our results.

Keywords Dynamic equations · Local and nonlocal conditions · Fixed point theorems · Measure of noncompactness

Mathematics Subject Classification 34N05 · 34A12 · 47H10 · 47H08

1 Introduction

Dynamic equations on time scales play a significant role in the mathematical modelling of numerous real-world phenomena involving continuous and discrete data simultaneously, for example, in population dynamics [20, 37], in economics [3, 4], in control theory [24, 34], and in optimization [36]. In recent years, the theory of dynamic equations on time scales has been extensively investigated by several researchers. The sphere of study of dynamic equations covers various aspects like qualitative and

✉ Sanket Tikare
sanket.tikare@gmail.com; sankettikare@rjcollege.edu.in

Martin Bohner
bohner@mst.edu

Bipan Hazarika
bh_rgu@yahoo.co.in; bh_gu@gauhati.ac.in

Ravi P. Agarwal
ravi.agarwal@tamuk.edu

[1] Department of Mathematics, Ramniranjan Jhunjhunwala College, Mumbai, Maharashtra 400 086, India

[2] Department of Mathematics and Statistics, Missouri S&T, Rolla, MO 65409-0020, USA

[3] Department of Mathematics, Gauhati University, Guwahati, Assam 781014, India

[4] Department of Mathematics, Texas A&M University-Kingsville, Kingsville, TX 78363-8202, USA

quantitative properties of solutions, stability of solutions, controllability of solutions, and applications in various areas of applied science and engineering [5, 17, 18, 22, 24, 29, 36], to mention a few.

It is well known that dynamic equations on time scales are an excellent tool for modelling any real-world phenomena that contain discrete and continuous-time data simultaneously. Motivated by numerous applications of dynamic equations in various areas of applied science, engineering, and technology, in this paper, we study the existence of solutions of two classes of dynamic first-order initial value problems. We first discuss the existence of solutions to the dynamic first-order local initial value problem

$$\begin{cases} x^{\Delta} + p(t)x^{\sigma} = f(t,x), & t \in \mathcal{I}^{\kappa}; \\ x(0) = A, \end{cases} \tag{1}$$

where $A \in X$ is a given constant.

It is well known that, in various modelling, equations coupled with nonlocal conditions give better results than those with local conditions. Also, in [2, 10, 11, 31], the authors studied some classes of nonlocal initial value problems for dynamic equations. Motivated by the work of the above papers, we next discuss the existence of solutions of the dynamic first-order nonlocal initial value problem

$$\begin{cases} x^{\Delta} + p(t)x^{\sigma} = f(t,x), & t \in \mathcal{I}^{\kappa}; \\ x(0) = \Phi(x), \end{cases} \tag{2}$$

where $\Phi : \mathrm{C}(\mathcal{I}, X) \to X$ is continuous. Here, $\mathrm{C}(\mathcal{I}, X)$ denotes the family of continuous functions from $\mathcal{I}$ to X. In (1) and (2), x is the unknown function to be found, x^{Δ} represents the delta derivative of x, $x^{\sigma} = x \circ \sigma$, $f : \mathcal{I} \times X \to X$ may be a nonlinear function, $p : \mathcal{I} \to \mathbb{R}$ is regressive and rd-continuous, X is a Banach space, and $\mathcal{I}^{\kappa} = \mathcal{I} \setminus (\rho(\sup \mathcal{I}), \sup \mathcal{I}]$ if $\sup \mathcal{I} < \infty$ otherwise $\mathcal{I}^{\kappa} = \mathcal{I}$.

In the literature, several methods have been employed to study the existence of solutions to dynamic equations on time scales. The approach of using fixed point theory is well known, for example, see [9, 11, 26, 27, 32, 33]. Also, the concept of measure of noncompactness has been successfully used to study the problem of existence of solutions for various integral, differential, and difference equations. Some of the related work can be observed in [13–16, 25, 28, 30]. The measure of noncompactness associates numbers to sets in such a way that compact sets all get the measure 0, and other sets get measures that are bigger according to "how far" they are removed from compactness. Darbo, in [12], first implemented the measure of noncompactness to generalize the Banach fixed point theorem for Banach spaces. The main advantage of using the measure of noncompactness is that the compactness of the domain of the operator has been relaxed to obtain the fixed point of an operator. In this paper, we will apply the Sadovskiĭ and Mönch fixed point theorems with the measure of noncompactness to prove the existence of solutions of problems (1) and (2). The class of equations in (1) and (2) is more general, and it can include several previously studied problems as special cases, [2, 9, 21, 33] to mention a few.

The paper is structured as follows. Section 2 comprises some fundamental definitions and results to follow the paper. Section 3 deals with our main results of existence of solutions. In Sect. 4, we provide some new examples to illustrate our results. Finally, Sect. 5 contains concluding remarks and some further research directions.

2 Preliminaries

In this section, we set forth some fundamental definitions and results needed for our subsequent discussion. We assume that the reader of this paper is familiar with basic concepts of time scales calculus, and for a review of the topic, we refer to [6, 7]. A time scale, denoted by $\mathbb{T}$, is a nonempty closed subset of $\mathbb{R}$. We assume that $0 \in \mathbb{T}$. For $T \in \mathbb{T}$ with $0 < T < \infty$, the time scale interval $\mathcal{I}$ is defined by $\mathcal{I} = [0, T]_{\mathbb{T}} := [0, T] \cap \mathbb{T} = \{t \in \mathbb{T} : 0 \le t \le T\}$.

Definition 1 (See [6, Definition 1.58]) A function $x : \mathbb{T} \to X$ is said to be rd-continuous if it is continuous at every right-dense points in $\mathbb{T}$ and its left sided limits exist at left dense points in $\mathbb{T}$. The notation $\mathrm{C}_{\mathrm{rd}}(\mathbb{T}, X)$ denotes the set of all rd-continuous functions $x : \mathbb{T} \to X$.

Definition 2 (See [19, Definition 5]) A function $f : \mathbb{T} \times X \to X$ is said to be rd-continuous on $\mathbb{T} \times X$ if $f(\cdot, x)$ is rd-continuous on $\mathbb{T}$ for each fixed $x \in X$ and $f(t, \cdot)$ is continuous on X for each fixed $t \in \mathbb{T}$. The notation $\mathrm{C}_{\mathrm{rd}}(\mathbb{T} \times X, X)$ denotes the set of all rd-continuous functions $f : \mathbb{T} \times X \to X$.

Definition 3 (See [6, Definition 2.25]) A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$, where the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. The notation $\mathcal{R}(\mathbb{T}, \mathbb{R})$ denotes the set of all regressive functions $p : \mathbb{T} \to \mathbb{R}$.

Definition 4 (See [6, Definition 2.30]) For a regressive function $p : \mathbb{T} \to \mathbb{R}$ and $t_0 \in \mathbb{T}$, the exponential function $e_p(\cdot, t_0)$ on the time scale $\mathbb{T}$ is defined as the unique solution of the initial value dynamic problem

$$x^{\Delta}(t) = p(t)x, \; x(t_0) = 1, \quad t \in \mathbb{T}^{\kappa}.$$

For $p, q \in \mathcal{R}(\mathbb{T}, \mathbb{R})$, we define

$$p \oplus q := p + q + \mu pq, \quad \ominus p := \frac{-p}{1 + \mu p}, \quad p \ominus q := p \oplus (\ominus q).$$

Some fundamental properties of the exponential function are stated below.

Theorem 1 (See [6, Theorem 2.36]) *Assume that* $p, q : \mathbb{T} \to \mathbb{R}$ *are regressive and rd-continuous. Then the following hold.*

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
(iii) $1/e_p(t, s) = e_{\ominus p}(t, s)$;
(iv) $e_p(t, s) = 1/e_p(s, t)$;
(v) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
(vi) $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$;
(vii) $e_p(t, s)/e_q(t, s) = e_{p \ominus q}(t, s)$.

Throughout the paper, we denote

$$E := \sup_{s,t\in\mathcal{I}} |e_{\ominus p}(t,s)|.$$

Let $(X, \|\cdot\|_X)$ be a given Banach space. By $\mathrm{C}(\mathcal{I}, X)$, we denote the family of all continuous functions from $\mathcal{I}$ into X, which is a Banach space coupled with the norm $\|\cdot\|$ defined as

$$\|x\| := \sup_{t\in\mathcal{I}} \|x(t)\|_X.$$

Definition 5 (See [35, Definition 11.1]) Let M be a bounded subset of a Banach space X. The Kuratowski measure of noncompactness of M, $\chi(M)$, is defined to be the infimum of all $\varepsilon > 0$ with the property that M can be covered by finitely many sets, each of whose diameter is less than or equal to ε. That is,

$$\chi(M) := \inf\{\varepsilon > 0 : M \text{ admits a finite covering by sets of diameter} \leq \varepsilon\}.$$

We list some properties of the measure of noncompactness.

Theorem 2 (See [35, Propositon 11.3]) *Assume that A and B are bounded subsets of a Banach space X and χ is the measure of noncompactness. Then we have the following.*

(i) *If $A \subset B$, then $\chi(A) \leq \chi(B)$;*
(ii) $\chi(A) = \chi(\overline{A})$, *where $\overline{A}$ denotes the closure of A;*
(iii) $\chi(A) = 0$ *if and only if A is relatively compact;*
(iv) $\chi(A \cup B) = \max\{\chi(A), \chi(B)\}$;
(v) $\chi(\alpha A) = |\alpha|\chi(A)\,(\alpha \in \mathbb{R})$;
(vi) $\chi(A + B) \leq \chi(A) + \chi(B)$;
(vii) $\chi(\operatorname{conv} A) = \chi(A)$, *where* $\operatorname{conv}(A)$ *denotes the convex extension of A;*
(viii) $\chi(A) \leq \operatorname{diam}(A)$.

The next lemma from [1] is stated in the context of time scales.

Lemma 1 (See [21, Lemma 2.7]) *Let $H \subset \mathrm{C}(\mathcal{I}, X)$ be a family of strongly equicontinuous functions. Let $H(t) := \{h(t) \in X : h \in H\}$ for $t \in \mathcal{I}$. Then*

$$\chi_{\mathrm{C}}(H) = \sup_{t\in\mathcal{I}} \chi(H(t)),$$

and the function $t \mapsto \chi(H(t))$ is continuous, where $\chi_{\mathrm{C}}(H)$ denotes the measure of noncompactness in $\mathrm{C}(\mathcal{I}, X)$.

Theorem 3 (Mean value theorem [9, Theorem 2.9]) *If $f : \mathcal{I} \to X$ is rd-continuous, then*

$$\int_{\mathcal{J}} f(t)\Delta t \in \mu_\Delta(J) \cdot \overline{\operatorname{conv}} f(\mathcal{J}),$$

where $\mathcal{J}$ is an arbitrary subinterval of $\mathcal{I}$ and $\mu_\Delta(\mathcal{J})$ is the Lebesgue delta-measure of $\mathcal{J}$.

Definition 6 (See [35, Definition 11.6]) Let X be a Banach space. A mapping $F : X \to X$ is said to be condensing if and only if F is bounded and continuous, and $\chi(F(B)) < \chi(B)$ for all bounded sets B in X with $\chi(B) > 0$, where χ is the measure of noncompactness.

In the existence result for local initial value problem (1), we apply the fixed point theorem due to Sadovskiĭ, which is stated as follows.

Theorem 4 (See [35, Theorem 11.A]) *Let M be a nonempty, closed, bounded, and convex subset of a Banach space X. Then the condensing map $F : M \to M$ has a fixed point in M.*

The following fixed point theorem due to Mönch will be used for the existence result of nonlocal initial value problem (2).

Theorem 5 (See [23, Theorem 2.1]) *Let D be a closed and convex subset of a Banach space X. Let $F : D \to D$ be a continuous mapping with the property that there exists $x \in D$ such that for any countable set C of D satisfying that $\overline{C} = \overline{\mathrm{conv}}(F(C) \cup \{x\})$, we have that C is a relatively compact D. Then F has a fixed point in D.*

3 Main results

The following lemma is proved in [8, Lemma 3.1], which establishes the equivalence of dynamic problem (1) and a delta integral equation.

Lemma 2 *Let $A \in X$, $p \in \mathcal{R}(\mathcal{I}, X)$. Assume that $f \in \mathrm{C_{rd}}(\mathcal{I} \times X, X)$. Then, x is a solution of the dynamic problem (1) if and only if x satisfies the integral equation*

$$x(t) = e_{\ominus p}(t,0)A + \int_0^t e_{\ominus p}(t,s)f(s,x(s))\Delta s. \tag{3}$$

We can state a similar lemma for dynamic problem (2) as follows.

Lemma 3 *Let $\Phi : \mathrm{C}(\mathcal{I}, X) \to X$ be continuous and $p \in \mathcal{R}(\mathcal{I}, X)$. Assume that $f \in \mathrm{C_{rd}}(\mathcal{I} \times X, X)$. Then, x is a solution of the dynamic problem (2) if and only if x satisfies the integral equation*

$$x(t) = e_{\ominus p}(t,0)\Phi(x) + \int_0^t e_{\ominus p}(t,s)f(s,x(s))\Delta s. \tag{4}$$

In the following theorem, we obtain the existence of solutions of local initial value problem (1) applying Theorem 4.

Theorem 6 *Consider the dynamic problem (1). Let $f : \mathcal{I} \times X \to X$ be rd-continuous. Assume that the following hypotheses are satisfied.*

(H_1) *There exists a positive constant N such that*

$$\|f(t,u)\|_X \leq N(1+\|u\|_X) \tag{5}$$

for all $t \in \mathcal{I}$ *and each* $u \in X$.

(H$_2$) *There exists an rd-continuous function* $L : \mathcal{I} \times \mathbb{R}^+ \to \mathbb{R}^+$ *such that for each continuous function* $u : \mathbb{R}^+ \to \mathbb{R}^+$, $L(\cdot, u(\cdot))$ *is continuous on* $\mathcal{I}$ *and* $\int_0^T L(s,v)\Delta s < v$ *for each* $v > 0$.

(H$_3$) *For any compact subinterval* $\mathcal{J}$ *of* $\mathcal{I}$ *and each nonempty bounded subset* W *of* X, *and for all* $t \in \mathcal{I}$, *we have*

$$\chi(e_{\ominus p}(t,\mathcal{J})f(\mathcal{J}\times W)) \leq \sup_{s\in\mathcal{J}} L(s,\chi(W)).$$

Then the local initial value problem (1) *has at least one solution on* $\mathcal{I}$ *provided* $ENT < 1$.

Proof Let $r > 0$ be such that

$$\frac{E\|A\|_X + ENT}{1-ENT} \leq r \tag{6}$$

and consider the closed ball

$$B_r := \left\{x \in \mathrm{C}(\mathcal{I},X) : \ \|x\| \leq r\right\}.$$

The set B_r is a bounded, closed, and convex subset of $\mathrm{C}(\mathcal{I},X)$. We see that the set B_r is an equicontinuous subset of $\mathrm{C}(\mathcal{I},X)$. To this end, let $t', t'' \in \mathcal{I}$ with $t' \leq t''$. Then, from (3), we have

$$\begin{aligned}
&\|x(t'') - x(t')\|_X \\
&\quad = \left\| e_{\ominus p}(t'',0)A + \int_0^{t''} e_{\ominus p}(t'',s)f(s,x(s))\Delta s - e_{\ominus p}(t',0)A \right. \\
&\qquad \left. - \int_0^{t'} e_{\ominus p}(t',s)f(s,x(s))\Delta s \right\|_X \\
&\quad \leq |e_{\ominus p}(t'',0) - e_{\ominus p}(t',0)|\,\|A\|_X \\
&\quad + \left\| \int_0^{t''} e_{\ominus p}(t'',s)f(s,x(s))\Delta s - \int_0^{t'} e_{\ominus p}(t',s)f(s,x(s))\Delta s \right\|_X \\
&\quad = |e_{\ominus p}(t'',0) - e_{\ominus p}(t',0)|\,\|A\|_X + \left\| e_{\ominus p}(t'',0)\int_0^{t'} e_p(s,0)f(s,x(s))\Delta s \right. \\
&\quad \left. + e_{\ominus p}(t'',0)\int_{t'}^{t''} e_p(s,0)f(s,x(s))\Delta s - e_{\ominus p}(t',0)\int_0^{t'} e_p(s,0)f(s,x(s))\Delta s \right\|_X \\
&\quad \leq |e_{\ominus p}(t'',0) - e_{\ominus p}(t',0)|\,\|A\|_X \\
&\qquad + |e_{\ominus p}(t'',0) - e_{\ominus p}(t',0)| \int_0^{t'} |e_p(s,0)|\,\|f(s,x(s))\|_X\Delta s \\
&\qquad + |e_{\ominus p}(t'',0)| \int_{t'}^{t''} |e_p(s,0)|\,\|f(s,x(s))\|_X\Delta s.
\end{aligned}$$

A similar inequality is obtained for $t'' \leq t'$. Since $e_{\ominus p}(\cdot, 0)$ is continuous, the right-hand side of the above inequality tends to zero as $t'' - t' \to 0$. Thus, B_r is equicontinuous. Now, we define a mapping $F : B_r \to C(\mathcal{I}, X)$ by

$$F(x)(t) := e_{\ominus p}(t,0)A + \int_0^t e_{\ominus p}(t,s)f(s,x(s))\Delta s. \tag{7}$$

Let $x \in B_r$. Then, for $t \in \mathcal{I}$, we can write

$$\begin{aligned}
\|F(x)(t)\|_X &= \left\| e_{\ominus p}(t,0)A + \int_0^t e_{\ominus p}(t,s)f(s,x(s))\Delta s \right\|_X \\
&\leq |e_{\ominus p}(t,0)|\,\|A\|_X + \int_0^t |e_{\ominus p}(t,s)|\,\|f(s,x(s))\|_X \Delta s \\
&\leq E\|A\|_X + E\int_0^t \|f(s,x(s))\|_X \Delta s \\
&\leq E\|A\|_X + E\int_0^t N(1 + \|x\|_X)\Delta s \\
&\leq E\|A\|_X + EN(1+r)T \\
&\overset{(6)}{\leq} r.
\end{aligned}$$

Thus, $F(B_r)$ is bounded. Also, equicontinuity of $F(B_r)$ can be verified similarly to that of B_r. Hence $F(B_r) \subseteq B_r$ and $F : B_r \to B_r$ is a well-defined mapping. Next, we show that $F : B_r \to B_r$ is continuous. Let $\{x_n\}$ be a sequence of elements in B_r such that $x_n \to x$ in B_r. Then, for $t \in \mathcal{I}$, we compute

$$\begin{aligned}
\|F(x_n)(t) - F(x)(t)\|_X &= \Big\| e_{\ominus p}(t,0)A + \int_0^t e_{\ominus p}(t,s)f(s,x_n(s))\Delta s \\
&\qquad - e_{\ominus p}(t,0)A - \int_0^t e_{\ominus p}(t,s)f(s,x(s))\Delta s \Big\|_X \\
&= \left\| \int_0^t e_{\ominus p}(t,s)f(s,x_n(s))\Delta s - \int_0^t e_{\ominus p}(t,s)f(s,x(s))\Delta s \right\|_X \\
&\leq \int_0^t |e_{\ominus p}(t,s)|\,\|f(s,x_n(s)) - f(s,x(s))\|_X \Delta s.
\end{aligned}$$

By the continuity of f, we have $\|F(x_n)(t) - F(x)(t)\|_X \to 0$ as $n \to \infty$. This shows that the mapping $F : B_r \to B_r$ is continuous. Now, let D be a nonempty strongly equicontinuous subset of B_r. Then, by Lemma 1, we see that the map $t \mapsto \chi(D(t))$ is continuous on $\mathcal{I}$. Since B_r is bounded, D is also bounded. Let τ be a real number such that $0 < \tau \leq T$. By hypothesis (H_2), $(t,s) \mapsto L(t, \chi(D(s)))$ is continuous for $(t,s) \in \mathcal{I} \times \mathbb{R}^+$. Therefore, for given $\varepsilon > 0$, there exists $\delta > 0$ such that for $t', t'' \in [0,\tau]_{\mathbb{T}}$ with $|t' - t''| < \delta$, we have $|L(t', \chi(D(s))) - L(t'', \chi(D(s)))| < \varepsilon$. Define $I_i := [t_{i-1}, t_i]_{\mathbb{T}}$ and

$$W_i := \bigcup_{s \in I_i} D(s)$$

for $i = 1, 2, \ldots, n$, where $n \in \mathbb{N}$ and

$$t_i := \begin{cases} 0 & \text{if } i = 0, \\ \tau & \text{if } i = n, \\ \sup\{s \in [0,\tau]_{\mathbb{T}} : 0 < s - t_{i-1} < \delta\} & \text{if } i = 1, 2, \dots, n-1. \end{cases}$$

If $t_i = t_{i-1}$ for some $i, 1 \le i \le n$, then we set $t_{i+1} = \inf\{s \in \mathbb{T} : s > t_i\}$. Now, by Theorem 3, we see that

$$\begin{aligned} \int_0^\tau e_{\ominus p}(\tau,s) f(s,x(s))\Delta s &\in \sum_{i=1}^n \mu_\Delta(I_i)\overline{\mathrm{conv}}(e_{\ominus p}(\tau,s)f(s,x(s)) : s \in I_i) \\ &\subset \sum_{i=1}^n \mu_\Delta(I_i)\overline{\mathrm{conv}}(e_{\ominus p}(\tau,I_i)f(I_i \times W_i)) \quad \text{for } x \in D. \end{aligned}$$

Hence, for $\tau \in \mathcal{I}$,

$$\begin{aligned} \chi(F[D](\tau)) &= \chi\left\{ e_{\ominus p}(\tau,0)A + \int_0^\tau e_{\ominus p}(\tau,s)f(s,x(s))\Delta s : x \in D \right\} \\ &\le \chi(e_{\ominus p}(\tau,0)A) + \chi\left\{ \int_0^\tau e_{\ominus p}(\tau,s)f(s,x(s))\Delta s : x \in D \right\} \\ &= \chi\left\{ \int_0^\tau e_{\ominus p}(\tau,s)f(s,x(s))\Delta s : x \in D \right\} \\ &= \chi\left(\int_0^\tau e_{\ominus p}(\tau,s)f(s,D(s))\Delta s \right) \\ &\le \chi\left(\sum_{i=1}^n \mu_\Delta(I_i)\overline{\mathrm{conv}}(e_{\ominus p}(\tau,I_i)f(I_i \times W_i)) \right) \\ &\le \sum_{i=1}^n |\mu_\Delta(I_i)|\chi(\overline{\mathrm{conv}}(e_{\ominus p}(\tau,I_i)f(I_i \times W_i))). \end{aligned}$$

By hypothesis (H_3), we can write

$$\chi(F[D](\tau)) \le \sum_{i=1}^n \mu_\Delta(I_i) \sup_{s \in I_i} L(s, \chi(W_i)).$$

Let $u \in I_i$ be such that $L(u, \chi(W_i)) = \sup_{s \in I_i} L(s, \chi(W_i))$, where

$$W_i = \bigcup_{u \in I_i} D(u).$$

Then, we obtain

$$\begin{aligned}\chi(F[D](\tau)) &\le \sum_{i=1}^{n} \mu_{\Delta}(I_i)\, L(u_i, \chi(W_i)) \\ &= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} L(u_i, \chi(W_i))\Delta s \\ &= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (L(u_i, \chi(W_i)) - L(s, \chi(W_i)) + L\big(s, \chi(W_i)))\Delta s \\ &\le \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (L(s, \chi(W_i)) + |L(s, \chi(W_i)) - L(u_i, \chi(W_i))|)\Delta s \\ &\le \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} L(s, \chi(W_i))\Delta s + T\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\begin{aligned}\chi(F[D](\tau)) &\le \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} L(s, \chi(W_i))\Delta s \\ &= \int_{0}^{\tau} L(s, \chi(W_i))\Delta s \\ &\le \int_{0}^{T} L\left(s, \sup_{u\in I_i} \chi(D(u))\right)\Delta s.\end{aligned}$$

Therefore,

$$\sup_{\tau\in\mathcal{I}} \chi(F[D](\tau)) \le \int_{0}^{T} L\left(s, \sup_{u\in I_i} \chi(D(u))\right)\Delta s.$$

But, from hypothesis (H_2), we have

$$\int_{0}^{T} L\left(s, \sup_{u\in I_i} \chi(D(u))\right)\Delta s < \sup_{u\in I_i} \chi(D(u)).$$

This yields

$$\sup_{\tau\in\mathcal{I}} \chi(F[D](\tau)) < \sup_{u\in I_i} \chi(D(u)).$$

By Lemma 1, we can write

$$\chi_{C}(F[D]) < \chi_{C}(D)$$

for any nonempty bounded subset D of B_r with $\chi_{C}(D) > 0$. Thus, F is a condensing map according to Definition 6, and by Theorem 4, we can conclude that F has at least one fixed point in B_r. This completes the proof. □

Remark 1 Theorem 6 coincides with the results given in [21] with $T = \infty$ and $p(t) \equiv 0$.

As an immediate result of Theorem 6, we can obtain the following corollary.

Corollary 1 *Let $f : \mathcal{I} \times X \to X$ be rd-continuous such that there exist two bounded functions $\psi_1, \psi_2 : \mathcal{I} \to \mathbb{R}_+$ with*

$$\|f(t,u)\|_X \le \psi_1(t) + \psi_2(t)\|u\|_X$$

for all $t \in \mathcal{I}$ and $u \in X$. Suppose that the hypotheses (H_2) and (H_3) hold. Then, the dynamic problem (1) has at least one solution if $EM_2T < 1$ for some $M_2 \in \mathbb{R}^+$.

Proof Since ψ_1, ψ_2 are bounded on $\mathcal{I}$, there exist $M_1, M_2 \in \mathbb{R}^+$ such that $\psi_1(t) \le M_1$ and $\psi_2(t) \le M_2$ for all $t \in \mathcal{I}$. Then

$$\begin{aligned}\|f(t,u)\|_X &\le \psi_1(t) + \psi_2(t)\|u\|_X \\ &\le M_2\left(\frac{M_1}{M_2} + \|u\|_X\right).\end{aligned}$$

We choose $r > 0$ such that $\dfrac{E\|A\|_X + EM_1T}{1 - EM_2T} \le r$. Now, applying Theorem 6, we get the required result. □

In the next theorem, we obtain the existence of solutions of nonlocal initial value problem (2) applying Theorem 5.

Theorem 7 *Consider the dynamic problem (2). Let $f : \mathcal{I} \times X \to X$ be rd-continuous. Assume that the following hypotheses are satisfied.*

(H_4) *There exists a positive constant N such that*

$$\|f(t,u)\|_X \le N(1 + \|u\|_X) \tag{8}$$

for all $t \in \mathcal{I}$ and each $u \in X$.

(H_5) *There exists a positive constant Q such that*

$$\|\Phi(u)\|_X \le Q(1 + \|u\|_X) \tag{9}$$

for each $u \in X$.

(H_6) *For each bounded subset Y of X, there exists $\alpha \in (0, 1/T)$ such that*

$$\chi(e_{\ominus p}(\mathcal{J}, \mathcal{J})f(\mathcal{J}, Y)) \le \alpha\chi(Y)$$

for each subinterval $\mathcal{J}$ of $\mathcal{I}$, and for each bounded subset W of X, we have

$$\chi(\Phi(W)) \le Q\chi(W).$$

If $EQ + T\alpha < 1$ and $EN < \alpha$, then the dynamic nonlocal initial value problem (2) has at least one solution on $\mathcal{I}$.

Proof Let $r > 0$ be such that

$$\frac{EQ + ENT}{1 - (EQ + ENT)} \leq r \tag{10}$$

and consider the closed ball

$$B_r := \{x \in \mathrm{C}(\mathcal{I}, X) : \|x\| \leq r\}.$$

The set B_r is a closed, convex, and equicontinuous subset of $\mathrm{C}(\mathcal{I}, X)$. This can be seen from the proof of Theorem 6. Define the mapping $F : B_r \to \mathrm{C}(\mathcal{I}, X)$ by

$$F(x)(t) := e_{\ominus p}(t, 0)\Phi(x) + \int_0^t e_{\ominus p}(t, s)f(s, x(s))\Delta s. \tag{11}$$

Then, by Lemma 3, the fixed points of the map F are the solutions of (2). For $x \in B_r$ and $t \in \mathcal{I}$, from (11), we obtain

$$\begin{aligned}
\|F(x)(t)\|_X &= \left\| e_{\ominus p}(t, 0)\Phi(x) + \int_0^t e_{\ominus p}(t, s)f(s, x(s))\Delta s \right\|_X \\
&\leq |e_{\ominus p}(t, 0)| \|\Phi(x)\|_X + \left\| \int_0^t e_{\ominus p}(t, s)f(s, x(s))\Delta s \right\|_X \\
&\leq EQ(1 + \|x\|_X) + E\int_0^t N(1 + \|x\|_X)\Delta s \\
&\leq EQ(1 + r) + EN(1 + r)t \\
&\leq EQ(1 + r) + EN(1 + r)T \\
&\overset{(10)}{\leq} r.
\end{aligned}$$

This implies that $F(x) \in B_r$ for all $x \in B_r$. Hence, $F(B_r) \subset B_r$. Therefore, F maps B_r into itself. Now, let $\{x_n\}$ be a sequence in B_r such that $\|x_n - x\| \to 0$. Then, for each $t \in \mathcal{I}$,

$$\begin{aligned}
\|F(x_n)(t) - F(x)(t)\|_X &= \Bigg\| e_{\ominus p}(t, 0)\Phi(x_n) + \int_0^t e_{\ominus p}(t, s)f(s, x_n(s))\Delta s \\
&\qquad - e_{\ominus p}(t, 0)\Phi(x) - \int_0^t e_{\ominus p}(t, s)f(s, x(s))\Delta s \Bigg\|_X \\
&\leq |e_{\ominus p}(t, 0)| \|\Phi(x_n) - \Phi(x)\|_X \\
&\quad + \int_0^t |e_{\ominus p}(t, s)| \|f(s, x_n(s)) - f(s, x(s))\|_X \Delta s \\
&\leq E\|\Phi(x_n) - \Phi(x)\|_X \\
&\quad + E\int_0^t \|f(s, x_n(s)) - f(s, x(s))\|_X \Delta s.
\end{aligned}$$

Since $f \in \mathrm{C}_{\mathrm{rd}}(\mathcal{I} \times X, X)$ and Φ is continuous on $\mathrm{C}(\mathcal{I}, X)$, we can deduce that $\|F(x_n) - F(x)\| \to 0$. Thus, F is continuous on B_r. Hence, $F : B_r \to B_r$ is a continuous map. Now, let R be a countable subset of B_r such that $\overline{R} = \overline{\mathrm{conv}}(\{x\} \cup F(R))$ for some $x \in B_r$. The set R is a countable subset of the bounded and equicontinuous set B_r. So, it is bounded and equicontinuous. Therefore, the function $t \mapsto v(t) = \chi(R(t))$ is continuous on $\mathcal{I}$, where $R(t) := \{v(t) \in X : v \in R\}$ for $t \in \mathcal{I}$. Let

$$F(R)(t) = \left\{ e_{\ominus p}(t,0)\Phi(x) + \int_0^t e_{\ominus p}(t,s)f(s,x(s))\Delta s : \; x \in R, \; t \in \mathcal{I} \right\}.$$

Then,

$$F(R)(t) = e_{\ominus p}(t,0)\Phi(R(t)) + \int_0^t e_{\ominus p}(t,s)f(s,R(s))\Delta s.$$

By properties of measure of noncompactness χ and hypotheses, for each $t \in \mathcal{I}$, we have

$$\begin{aligned}
v(t) &\le \chi(F(R(t)) \cup \{x\}) \\
&\le \chi(F(R(t))) \\
&\le \chi\left(e_{\ominus p}(t,0)\Phi(R(t)) + \int_0^t e_{\ominus p}(t,s)f(s,R(s))\Delta s \right) \\
&\le \chi(e_{\ominus p}(t,0)\Phi(R(t))) + \left(\int_0^t e_{\ominus p}(t,s)f(s,R(s))\Delta s \right) \\
&\le |e_{\ominus p}(t,0)|\chi(\Phi(R(t))) \\
&\quad + \chi\left(\mu_\Delta([0,t]_{\mathbb{T}})\overline{\text{conv}}\left(e_{\ominus p}([0,t]_{\mathbb{T}},[0,t]_{\mathbb{T}})f([0,t]_{\mathbb{T}},R([0,t]_{\mathbb{T}}))\right)\right) \\
&\le EQ\chi(R(t)) + t\chi\left(\overline{\text{conv}}(e_{\ominus p}([0,t]_{\mathbb{T}},[0,t]_{\mathbb{T}})f([0,t]_{\mathbb{T}},R([0,t]_{\mathbb{T}})))\right) \\
&\le EQ\chi(R(t)) + t\chi\left(\text{conv}\,(e_{\ominus p}([0,t]_{\mathbb{T}},[0,t]_{\mathbb{T}})f([0,t]_{\mathbb{T}},R([0,t]_{\mathbb{T}})))\right) \\
&\le EQ\chi(R(t)) + T\chi\left(e_{\ominus p}([0,t]_{\mathbb{T}},[0,t]_{\mathbb{T}})f([0,t]_{\mathbb{T}},R([0,t]_{\mathbb{T}}))\right) \\
&\le EQ\chi(R(t)) + T\alpha\chi(R(t)) \\
&= (EQ + T\alpha)\chi(R(t)).
\end{aligned}$$

That is,

$$\chi(R(t)) \le (EQ + T\alpha)\chi(R(t)).$$

This gives

$$(1 - (EQ + T\alpha))\chi(R(t)) \le 0.$$

But, by assumption, $1 - (EQ + T\alpha) > 0$. Hence $\chi(R(t)) = 0$. Therefore, $R(t)$ is relatively compact in B_r. Now, applying Theorem 5, we conclude that the mapping F has a fixed point in B_r. Hence, the dynamic initial value problem (2) has a solution in $\mathcal{I}$. □

Remark 2 Theorem 7 also holds even if the condition '$u \in X$' in the hypotheses (H_4) and (H_5) are replaced by the local condition '$u \in B^*_\rho$', where

$$B^*_\rho := \{x \in X : \; \|x\|_X \le \rho\} \text{ for some } \rho > 0.$$

Remark 3 Since $\mu(t) < T$ for all $\mathcal{I}$, we may replace (5) and (8) by the growth condition

$$\left\|f(t,u)\right\|_X \le N\left(1 + \mu(t) + \|u\|_X\right) \tag{12}$$

and (9) by the growth condition

$$\|\Phi(u)\|_X \le Q(1+\mu(t)+\|u\|_X). \tag{13}$$

Then, in the proof of Theorem 6, we obtain

$$\|F(x)(t)\|_X \le E\|A\|_X + EN(1+T+r)T,$$

and we choose $r>0$ such that $\dfrac{E\|A\|_X + ENT + ENT^2}{1-ENT} \le r$. In the proof of Theorem 7, we obtain

$$\|F(x)(t)\|_X \le EQ(1+T+r) + EN(1+T+r)T,$$

and we choose $r>0$ such that $\dfrac{(EQ+ENT)(1+T)}{1-(EQ-ENT)} \le r$. Results similar to Theorem 6 and 7 can be obtained without much change.

4 Examples

Now, in this section, we provide some new examples to illustrate our results. For simplicity, we assume $X=\mathbb{R}$.

Example 1 Let $\mathbb{T} := [0,1]\cup[2,3]$ and consider the dynamic initial value problem

$$\begin{cases} x^\Delta + p(t)x^\sigma = f\big(t,x\big), & t\in\mathcal{I}^\kappa := [0,3]^\kappa_{\mathbb{T}};\\ x(0)=A, \end{cases} \tag{14}$$

where $f(t,x)=\frac{1}{2}\sin t + xe^{-t}$, $p(t)=-1$, and $A\in\mathbb{R}$. We see that

$$|f(t,x)| \le \left|\frac{1}{2}\sin t\right| + \left|xe^{-t}\right| < \frac{1}{2}(1+|x|)$$

and hence, the hypothesis (H_1) holds. Take $L(s,e^r)=\frac{s}{5}e^r$. Then hypothesis (H_2) is also satisfied because

$$\int_0^3 L(s,e^r)\Delta s < e^r.$$

Take $\mathcal{J}=[2,3]_{\mathbb{T}}$, and $W=[0,1]$. Then, we observe that $\sup_{s\in\mathcal{J}}L(s,\chi(W))=\frac{3}{5}$. Now, since for $\tau\in\mathcal{J}$ and $x\in W$, $f(\tau,x)<2$, we obtain

$$e_{\ominus(-1)}(t,\tau)f(\tau,x) < 2e_{\ominus(-1)}(t,\tau)$$

for $t\in\mathcal{I}$, $\tau\in\mathcal{J}$, and $x\in W$. But $\chi(e_{\ominus(-1)}(\mathcal{I},\mathcal{J})=0$. Therefore, we get $\chi(e_{\ominus(-1)}(\mathcal{I},\mathcal{J})f(\mathcal{J}\times W)=0$. Hence the hypothesis ($H_3$) also holds. Thus, the conclusion of Theorem 6 implies that the problem (14) has at least one solution on $[0,3]_{\mathbb{T}}$.

Example 2 Let $\mathbb{T}$ be any time scale with $\mu(t)>0$ and consider the dynamic initial value problem

$$\begin{cases} x^\Delta + p(t)x^\sigma = f\big(t,x\big), & t\in\mathbb{T}^\kappa;\\ x(0)=\Phi(x), \end{cases} \tag{15}$$

where $f(t,x) = \sin\left(\frac{1}{2+\cos t}\right) + 2x$, $\Phi(x) = \frac{3x}{1+\mu(t)}$, and $p(t) = \frac{1}{\mu(t)}$. We see that

$$|f(t,x)| \leq \left|\sin\left(\frac{1}{2+\cos t}\right)\right| + |2x| < 2(1+|x|)$$

and hence, the hypothesis (H_4) holds. Also,

$$|\Phi(x)| = \left|\frac{3x}{1+\mu(t)}\right| < 3(1+|x|).$$

Thus, hypothesis (H_5) holds. Take $W = [1,2]$ and $Y = [0,1]$. Then we find $\chi(\Phi(W)) \leq 3\Phi(W)$. Also, for $t \in \mathcal{J}$ and $x \in Y$, $f(t,x) < 3$. Next, for $s, t \in \mathcal{J}$ and $x \in Y$,

$$e_{\ominus p}(s,t) f(t,x) \leq 3e_{\ominus p}(s,t) < 3e^{qt} \quad \text{for } q > \frac{-1}{2\mu}.$$

But $\chi(e^{q(\mathcal{J})}) = 0$. So $\chi(e_{\ominus p}(\mathcal{J},\mathcal{J}) f(\mathcal{J},Y)) \leq \alpha$, $\alpha \in (0,1/T)$. This yields that the hypothesis (H_6) holds for these f and Φ. Consequently, Theorem 7 implies that the problem (15) has at least one solution on $\mathbb{T}$.

5 Conclusion

The results presented in this paper are essentially new in the context of time scales. Within this scope, they form a basis for the study of other dynamic problems such as dynamic inclusions and higher-order dynamic equations. By employing the simple useful formula $x^\sigma = x + \mu x^\Delta$, the interested reader can acquire various qualitative properties of dynamic equations with local as well as nonlocal conditions. Also, as a continuation of this work, employing the approach of measure of noncompactness, the other aspects of solutions, like monotonicity, periodicity, stability, attractivity, asymptotic behaviour, oscillations, and controllability for these dynamic problems can be studied in the near future. The present results can also be generalized by replacing the compactness conditions in the hypotheses (H_3) and (H_5) with general conditions. Of course, one can replace the Kuratowski measure of noncompactness χ with some other axiomatic measures of noncompactness.

Acknowledgements We are thankful to all four referees for their valuable suggestions and useful comments towards the improvement of the paper. The research work of Sanket Tikare is supported by the SEED grant of Ramniranjan Jhunjhunwala College (Ref. No. 935-17102019).

References

1. Ambrosetti, A.: Un teorema di esistenza per le equazioni differenziali negli spazi di Banach. Rend. Sem. Mat. Univ. Padova **39**, 349–361 (1967)
2. Anderson, D.R., Abdelkader, B.: Nonlocal initial value problem for first-order dynamic equations on time scales. Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **16**(Differential Equations and Dynamical Systems, suppl. S1), 222–226 (2009)
3. Atici, F.M., Biles, D.C., Lebedinsky, A.: An application of time scales to economics. Math. Comput. Model. **43**(7–8), 718–726 (2006)
4. Atici, F.M., Uysal, F.: A production-inventory model of HMMS on time scales. Appl. Math. Lett. **21**(3), 236–243 (2008)

5. Bohner, M., Martynyuk, A.A.: Elements of stability theory of A. M. Liapunov for dynamic equations on time scales. Nonlinear Dyn. Syst. Theory **7**(3), 225–251 (2007)
6. Bohner, M., Peterson, A.: Dynamic equations on time scales. An introduction with applications. Birkhäuser Boston, Inc., Boston, MA (2001)
7. Bohner, M., Peterson, A.: Advances in Dynamic Equations on Time Scales. Birkhäuser Boston Inc., Boston (2003)
8. Bohner, M., Tikare, S., dos Santos, I.L.D.: First-order nonlinear dynamic initial value problems. Int. J. Dyn. Syst. Differ. Equ. **11**(3/4), 241–254 (2021)
9. Cichoń, M., Kubiaczyk, I., Sikorska-Nowak, A., Yantir, A.: Existence of solutions of the dynamic Cauchy problem in Banach spaces. Demo. Math. **45**(3), 561–573 (2012)
10. Cichoń, M., Satco, B., Sikorska-Nowak, A.: Impulsive nonlocal differential equations through differential equations on time scales. Appl. Math. Comput. **218**(6), 2449–2458 (2011)
11. Dai, Q., Tisdell, C.C.: Existence of solutions to first-order dynamic boundary value problems. Int. J. Difference Equ. **1**(1), 1–17 (2006)
12. Darbo, G.: Punti uniti in trasformazioni a codominio non compatto. Rendiconti del Seminario matematico della Università di Padova **24**, 84–92 (1955)
13. Das, A., Hazarika, B., Arab, R., Mursaleen, M.: Applications of a fixed point theorem to the existence of solutions to the nonlinear functional integral equations in two variables. Rend. Circ. Mat. Palermo (2) **68**(1), 139–152 (2019)
14. Gronek, T., Schmeidel, E.: Existence of bounded solution of Volterra difference equations via Darbo‘s fixed-point theorem. J. Difference Equ. Appl. **19**(10), 1645–1653 (2013)
15. Hazarika, B., Arab, R., Mursaleen, M.: Applications of measure of noncompactness and operator type contraction for existence of solution of functional integral equations. Complex Anal. Oper. Theory **13**(8), 3837–3851 (2019)
16. Hazarika, B., Arab, R., Nashine, H.K.: Applications of measure of non-compactness and modified simulation function for solvability of nonlinear functional integral equations. Filomat **33**(17), 5427–5439 (2019)
17. Hilger, S.: An application of calculus on measure chains to Fourier theory and Heisenberg’s uncertainty principle. In:Honour of Professor Allan Peterson on the occasion of his 60th birthday, vol. 8, pp. 897–936. (2002)
18. Hoffacker, J., Tisdell, C.C.: Stability and instability for dynamic equations on time scales. Comput. Math. Appl. **49**(9–10), 1327–1334 (2005)
19. Karpuz, B.: On the existence and uniqueness of solutions to dynamic equations. Turk. J. Math. **42**(3), 1072–1089 (2018)
20. Kaufmann, E.R.: A Kolmogorov predator-prey system on a time scale. Dyn. Systems Appl. **23**(4), 561–573 (2014)
21. Kubiaczyk, I., Sikorska-Nowak, A.: Existence of solutions of the dynamic Cauchy problem on infinite time scale intervals. Discuss. Math. Differ. Incl. Control Optim. **29**, 113–126 (2009)
22. Malik, M., Kumar, V.: Controllability of neutral differential equation with impulses on time scales. Diff. Equ. Dyn. Syst pp. 1–15 (2019)
23. Mönch, H.: Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces. Nonlinear Anal. **4**(5), 985–999 (1980)
24. Naidu, D.S.: Singular perturbations and time scales in control theory and applications: an overview. Singular. Perturb. Dyn. Syst. Control Technol. **9**, 233–278 (2002)
25. Rabbani, M., Arab, R., Hazarika, B.: Solvability of nonlinear quadratic integral equation by using simulation type condensing operator and measure of noncompactness. Appl. Math. Comput. **349**, 102–117 (2019)
26. Santos, I.L.D.: Discontinuous dynamic equations on time scales. Rend. Circ. Mat. Palermo (2) **64**(3), 383–402 (2015)
27. Santos, I.L.D.: On qualitative and quantitative results for solutions to first-order dynamic equations on time scales. Bol. Soc. Mat. Mex. (3) **21**(2), 205–218 (2015)
28. Schmeidel, E., Zbąszyniak, Z.: An application of Darbo‘s fixed point theorem in the investigation of periodicity of solutions of difference equations. Comput. Math. Appl. **64**(7), 2185–2191 (2012)
29. Shen, Y.: The Ulam stability of first order linear dynamic equations on time scales. Results Math. **72**(4), 1881–1895 (2017)
30. Srivastava, H.M., Das, A., Hazarika, B., Mohiuddine, S.A.: Existence of solutions of infinite systems of differential equations of general order with boundary conditions in the spaces c_0 and ℓ_1 via the measure of noncompactness. Math. Methods Appl. Sci. **41**(10), 3558–3569 (2018)
31. Tikare, S.: Nonlocal initial value problems for first-order dynamic equations on time scales. Appl. Math. E Notes **21**, 410–420 (2021)

32. Tikare, S., Santos, I.L.D.: Existence results for solutions to discontinuous dynamic equations on time scales. Differ. Equ. Appl. **12**(1), 89–100 (2020)
33. Tisdell, C.C., Zaidi, A.: Basic qualitative and quantitative results for solutions to nonlinear, dynamic equations on time scales with an application to economic modelling. Nonlinear Anal. **68**(11), 3504–3524 (2008)
34. Wintz, N.J.: The Kalman filter on time scales. ProQuest LLC, Ann Arbor, MI, 2009. In: Thesis (Ph.D.)–Missouri University of Science and Technology
35. Zeidler, E.: Nonlinear functional analysis and its applications. I. Springer-Verlag, New York. Fixed-point theorems. Translated from the German by P.R. Wadsack (1986)
36. Zhu, Y., Jia, G.: Linear feedback of mean-field stochastic linear quadratic optimal control problems on time scales. Math. Probl. Eng. pages Art. ID 8051918, 11 (2020)
37. Zhuang, K.: Periodic solutions for a stage-structure ecological model on time scales. Electron. J. Diff. Equ. pp. No. 88, 7 (2007)