



Even-order half-linear advanced differential equations: improved criteria in oscillatory and asymptotic properties



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ABSTRACT

We establish some new criteria for oscillation and asymptotic behavior of solutions of even-order half-linear advanced differential equations. We study the case of canonical and the case of noncanonical equations subject to various conditions.

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1. Introduction

In the natural sciences, technology, and population dynamics, differential equations find many application fields; see [13]. In recent years, there has been an increasing interest in studying oscillation of various classes of differential equations. We refer the reader to [1–12,14–24] and the references cited therein.

Many authors studied oscillatory behavior of the higher-order differential equation

$$\left((x^{(n-1)}(t))^\alpha\right)' + f(x(\tau(t))) = 0.$$

As a special case

$$\left((x^{(n-1)}(t))^\alpha\right)' + q(t)x^\alpha(\tau(t)) = 0, \quad (1.1)$$

where n is even, α is the ratio of odd positive integers, Agarwal and Grace [2] and Agarwal et al. [5] established the following results.

Theorem 1.1 (See [2, Theorem 3.6]). *Let*

$$q, \tau \in C([t_0, \infty), \mathbb{R}), \quad q(t) \geq 0, \quad \text{and} \quad \tau(t) \geq t \text{ for } t \geq t_0. \quad (1.2)$$

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If

$$\int_t^\infty q(s)ds < \infty,$$

$$\liminf_{t \rightarrow \infty} \int_t^{\tau(t)} s^{n-2} \left(\int_s^\infty q(u)du \right)^{1/\alpha} ds > \frac{(n-2)!}{e},$$

and

$$\liminf_{t \rightarrow \infty} \int_t^{\tau(t)} [\tau(s) - s]^{n-2} \left(\int_{\tau(s)}^\infty q(u)du \right)^{1/\alpha} ds > \frac{(n-2)!}{e},$$

then (1.1) is oscillatory.

Theorem 1.2 (See [2, Theorem 4.1]). Let (1.2) hold. If for all constants $\theta \in (0, 1)$,

$$\limsup_{t \rightarrow \infty} \frac{t^{n-1}}{(n-1)!} \left[\int_t^\infty q(s)ds + \frac{\theta\alpha}{2(n-2)!} \int_t^\infty s^{n-2} \left(\int_s^\infty q(v)dv \right)^{(\alpha+1)/\alpha} ds \right]^{1/\alpha} > 1,$$

then (1.1) is oscillatory.

Theorem 1.3 (See [5, Theorem 2.1, $\sigma(t) = t$]). Let (1.2) hold. If there exists a function $\rho \in C^1([t_0, \infty), (0, \infty))$ such that, for all constants $\theta \in (1, \infty)$,

$$\int_t^\infty \left[\rho(t)q(t) - \theta \frac{(2(n-2)!)^\alpha}{(\alpha+1)^{\alpha+1}} \frac{(\rho'(t))^{\alpha+1}}{(t^{n-2}\rho(t))^\alpha} \right] dt = \infty,$$

then (1.1) is oscillatory.

Theorem 1.4 (See [5, Theorem 2.3]). Let (1.2) hold, $\tau \in C^1([t_0, \infty), \mathbb{R})$, and $\tau'(t) > 0$ for $t \geq t_0$. If

$$\limsup_{t \rightarrow \infty} t^{\alpha(n-1)} \int_t^\infty q(s)ds > ((n-1)!)^\alpha,$$

then (1.1) is oscillatory.

Grace and Lalli [11] considered oscillation of an even-order equation

$$x^{(n)}(t) + q(t)x(\tau(t)) = 0, \tag{1.3}$$

and obtained the following result.

Theorem 1.5 (See [11, Theorems 2 and 3, $\sigma(t) = t$]). Let (1.2) hold. If there exists a function $\rho \in C^1([t_0, \infty), (0, \infty))$ such that

$$\int_t^\infty \left[\rho(t)q(t) - \frac{(n-1)!}{2^{3-2n}} \frac{(\rho'(t))^2}{t^{n-2}\rho(t)} \right] dt = \infty,$$

then (1.3) is oscillatory.

Following the papers [2,5,11], we are concerned with an advanced differential equation

$$(r(t)(x^{(n-1)}(t))^\alpha)' + q(t)x^\alpha(\tau(t)) = 0, \tag{1.4}$$

where $t \geq t_0$, α is the ratio of odd positive integers, $r \in C^1([t_0, \infty), \mathbb{R})$, $r(t) > 0$, $r'(t) \geq 0$, $q, \tau \in C([t_0, \infty), \mathbb{R})$, $q(t) \geq 0$, and $\tau(t) \geq t$. Similar as in the papers by Džurina and Kotorová [9] and Li et al. [16], Eq. (1.4) is called canonical if

$$\int_{t_0}^\infty \frac{dt}{r^{1/\alpha}(t)} = \infty, \tag{1.5}$$

whereas it is termed noncanonical in the case when

$$\int_{t_0}^\infty \frac{dt}{r^{1/\alpha}(t)} < \infty. \tag{1.6}$$

By a solution of (1.4) we mean a function $x \in C^{n-1}[T_x, \infty)$, $T_x \geq t_0$, which has the property $r(x^{(n-1)})^\alpha \in C^1[T_x, \infty)$ and satisfies (1.4) on $[T_x, \infty)$. We consider only those solutions x of (1.4) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$ and assume that (1.4) possesses such solutions. A solution of (1.4) is said to be oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$; otherwise, it is termed nonoscillatory. Eq. (1.4) is called oscillatory if all its solutions are oscillatory.

Zhang et al. [23,24] obtained some oscillation criteria for (1.4) in the case $\tau(t) < t$. The natural question now is: *is it possible to establish new oscillation and asymptotic criteria for (1.4) in the case where $\tau(t) \geq t$?* Our aim in this paper is to give an affirmative answer to this question. In what follows, all functional inequalities are assumed to hold eventually, that is, they are satisfied for all t large enough.

2. Main results

In this section, we will establish some oscillation criteria for (1.4). First, we give the following.

Lemma 2.1 (See [19]). *Let $f \in C^n([t_0, \infty), \mathbb{R}^+)$. If $f^{(n)}(t)$ is eventually of one sign for all large t , then there exist a $t_x \geq t_0$ and an integer $l, 0 \leq l \leq n$ with $n + l$ even for $f^{(n)}(t) \geq 0$, or $n + l$ odd for $f^{(n)}(t) \leq 0$ such that*

$$l > 0 \text{ yields } f^{(k)}(t) > 0 \text{ for } t \geq t_x, k = 0, 1, \dots, l - 1, \text{ and}$$

$$l \leq n - 1 \text{ yields } (-1)^{l+k} f^{(k)}(t) > 0 \text{ for } t \geq t_x, k = l, l + 1, \dots, n - 1.$$

Lemma 2.2 (See [3, Lemma 2.2.3]). *Assume that f is as in Lemma 2.1, $f^{(n)}(t)f^{(n-1)}(t) \leq 0$ for $t \geq t_x$, and $\lim_{t \rightarrow \infty} f(t) \neq 0$. Then for every constant $\lambda \in (0, 1)$, there exists a $t_\lambda \in [t_x, \infty)$ such that*

$$f(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} |f^{(n-1)}(t)|$$

holds on $[t_\lambda, \infty)$.

Now, we present the main results. For the sake of convenience, we use the notation $(\rho'(t))_+ := \max(0, \rho'(t))$, $(\delta'(t))_+ := \max(0, \delta'(t))$, and $R(t) := \int_t^\infty (1/r^{1/\alpha}(s)) ds$.

First, we consider the case where (1.5) holds.

Theorem 2.3. *Let $n \geq 4$ be even and (1.5) hold. Assume that there exist two functions $\rho, \delta \in C^1([t_0, \infty), (0, \infty))$ such that, for some constant $\lambda_0 \in (0, 1)$,*

$$\int^\infty \left[\rho(t)q(t) - \frac{((n-2)!)^\alpha}{(\alpha+1)^{\alpha+1}\lambda_0^\alpha} \frac{r(t)((\rho'(t))_+)^{\alpha+1}}{(t^{n-2}\rho(t))^\alpha} \right] dt = \infty \tag{2.1}$$

and, either

$$\int^\infty q(s) ds = \infty, \tag{2.2}$$

or

$$\int^\infty \eta^{n-4} \frac{[\int_\eta^\infty q(s) ds]^{1/\alpha}}{r^{1/\alpha}(\eta)} d\eta = \infty, \tag{2.3}$$

or

$$\int^\infty \left[\delta(t) \frac{\int_t^\infty (\eta-t)^{n-4} \frac{[\int_\eta^\infty q(s) ds]^{1/\alpha}}{r^{1/\alpha}(\eta)} d\eta}{(n-4)!} - \frac{((\delta'(t))_+)^2}{4\delta(t)} \right] dt = \infty. \tag{2.4}$$

Then (1.4) is oscillatory.

Proof. Assume that (1.4) has a nonoscillatory solution x . Without loss of generality, we may assume that x is eventually positive. It follows from (1.4) and Lemma 2.1 that there exist two possible cases for $t \geq t_1$ large enough:

- (1) $x(t) > 0, x'(t) > 0, x''(t) > 0, x^{(n-1)}(t) > 0, x^{(n)}(t) \leq 0, (r(x^{(n-1)})^\alpha)'(t) \leq 0;$
- (2) $x(t) > 0, x^{(j)}(t) > 0, x^{(j+1)}(t) < 0$ for every odd number $j \in \{1, 2, \dots, n-3\}, x^{(n-1)}(t) > 0, x^{(n)}(t) \leq 0,$ and $(r(x^{(n-1)})^\alpha)'(t) \leq 0.$

Assume that case (1) holds. We know that $\lim_{t \rightarrow \infty} x'(t) \neq 0$. By virtue of Lemma 2.2, for every constant $\lambda \in (0, 1)$ and for all large t , we have

$$x'(t) \geq \frac{\lambda}{(n-2)!} t^{n-2} x^{(n-1)}(t), \text{ by setting } f(t) := x'(t). \tag{2.5}$$

Now we introduce a Riccati substitution

$$u(t) := \rho(t) \frac{r(t)(x^{(n-1)}(t))^\alpha}{(x(t))^\alpha}, \quad t \geq t_1. \tag{2.6}$$

Then $u(t) > 0$ on $[t_1, \infty)$ and

$$\begin{aligned} u'(t) &= \rho'(t) \frac{r(t)(x^{(n-1)}(t))^\alpha}{(x(t))^\alpha} + \rho(t) \frac{(r(t)(x^{(n-1)}(t))^\alpha)'}{(x(t))^\alpha} - \rho(t) \frac{\alpha r(t)(x^{(n-1)}(t))^\alpha x'(t)}{(x(t))^{\alpha+1}} \\ &\leq -\rho(t) \frac{q(t)x^\alpha(\tau(t))}{(x(t))^\alpha} + \rho'(t) \frac{r(t)(x^{(n-1)}(t))^\alpha}{(x(t))^\alpha} - \frac{\lambda\alpha}{(n-2)!} t^{n-2} \rho(t) \frac{r(t)(x^{(n-1)}(t))^\alpha}{(x(t))^{\alpha+1}} \end{aligned}$$

due to (2.5). Note that $x' > 0$ and $\tau(t) \geq t$. By virtue of (2.6), we have

$$u'(t) \leq -\rho(t)q(t) + \frac{(\rho'(t))_+}{\rho(t)}u(t) - \frac{\lambda\alpha t^{n-2}}{(n-2)!(\rho(t)r(t))^{1/\alpha}}u^{(\alpha+1)/\alpha}(t). \tag{2.7}$$

Set

$$B := \frac{(\rho'(t))_+}{\rho(t)}, \quad A := \frac{\lambda\alpha t^{n-2}}{(n-2)!(\rho(t)r(t))^{1/\alpha}}, \quad v := u(t).$$

Using the inequality

$$-Av^{(\alpha+1)/\alpha} + Bv \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}, \quad A > 0,$$

we conclude that

$$\frac{(\rho'(t))_+}{\rho(t)}u(t) - \frac{\lambda\alpha t^{n-2}}{(n-2)!(\rho(t)r(t))^{1/\alpha}}u^{(\alpha+1)/\alpha}(t) \leq \frac{((n-2)!)^\alpha}{(\alpha+1)^{\alpha+1}\lambda^\alpha} \frac{r(t)((\rho'(t))_+)^{\alpha+1}}{(t^{n-2}\rho(t))^\alpha}.$$

Putting the resulting inequality into (2.7), we obtain

$$u'(t) \leq -\rho(t)q(t) + \frac{((n-2)!)^\alpha}{(\alpha+1)^{\alpha+1}\lambda^\alpha} \frac{r(t)((\rho'(t))_+)^{\alpha+1}}{(t^{n-2}\rho(t))^\alpha}.$$

This yields

$$\int_{t_1}^s \left[\rho(t)q(t) - \frac{((n-2)!)^\alpha}{(\alpha+1)^{\alpha+1}\lambda^\alpha} \frac{r(t)((\rho'(t))_+)^{\alpha+1}}{(t^{n-2}\rho(t))^\alpha} \right] dt \leq u(t_1)$$

for all large s and for every constant $\lambda \in (0, 1)$, which contradicts (2.1).

Assume that case (2) holds. Integrating (1.4) from t_1 to t , we conclude that

$$-r(t_1)(x^{(n-1)}(t_1))^\alpha + \int_{t_1}^t q(s)x^\alpha(\tau(s))ds \leq 0.$$

By virtue of $x' > 0$ and $\tau(t) \geq t$, we obtain

$$\int_{t_1}^t q(s)ds \leq \frac{r(t_1)(x^{(n-1)}(t_1))^\alpha}{x^\alpha(t_1)},$$

which contradicts (2.2). Integrating (1.4) from t to ∞ , we arrive at the inequality

$$-r(t)(x^{(n-1)}(t))^\alpha + \int_t^\infty q(s)x^\alpha(\tau(s))ds \leq 0.$$

It follows from $x' > 0$ and $\tau(t) \geq t$ that

$$-x^{(n-1)}(t) + \frac{x(t)}{r^{1/\alpha}(t)} \left[\int_t^\infty q(s)ds \right]^{1/\alpha} \leq 0. \tag{2.8}$$

Suppose first that $n = 4$. Integrating (2.8) from t_1 to t , we have

$$\int_{t_1}^t \frac{\left[\int_\eta^\infty q(s)ds \right]^{1/\alpha}}{r^{1/\alpha}(\eta)} d\eta \leq -\frac{x''(t_1)}{x(t_1)},$$

which contradicts (2.3) (where $n = 4$). Suppose now that $n \geq 6$. Integrating (2.8) from t to ∞ for a total of $(n - 4)$ times, we conclude that

$$-x'''(t) + \frac{\int_t^\infty (\eta - t)^{n-5} \left[\int_\eta^\infty q(s)ds \right]^{1/\alpha} d\eta}{(n-5)!} x(t) \leq 0.$$

Another integration from t_1 to t yields

$$\frac{\int_{t_1}^\infty (\eta - t_1)^{n-4} \left[\int_\eta^\infty q(s)ds \right]^{1/\alpha} d\eta}{(n-4)!} \leq -\frac{x''(t_1)}{x(t_1)},$$

which contradicts (2.3). Integrating (2.8) from t to ∞ for a total of $(n - 3)$ times, we have

$$x''(t) + \frac{\int_t^\infty (\eta - t)^{n-4} \left[\int_\eta^\infty q(s)ds \right]^{1/\alpha} d\eta}{(n-4)!} x(t) \leq 0. \tag{2.9}$$

Now, we define a Riccati substitution

$$w(t) := \delta(t) \frac{x'(t)}{x(t)}, \quad t \geq t_1. \tag{2.10}$$

Then $w(t) > 0$ for $t \geq t_1$ and

$$w'(t) = \delta'(t) \frac{x'(t)}{x(t)} + \delta(t) \frac{x''(t)x(t) - (x'(t))^2}{x^2(t)}.$$

It follows now from (2.9) and (2.10) that

$$w'(t) \leq -\delta(t) \frac{\int_t^\infty (\eta - t)^{n-4} \frac{[\int_\eta^\infty q(s)ds]^{1/\alpha}}{r^{1/\alpha}(\eta)} d\eta}{(n-4)!} + \frac{(\delta'(t))_+}{\delta(t)} w(t) - \frac{1}{\delta(t)} w^2(t). \tag{2.11}$$

Hence we have

$$w'(t) \leq -\delta(t) \frac{\int_t^\infty (\eta - t)^{n-4} \frac{[\int_\eta^\infty q(s)ds]^{1/\alpha}}{r^{1/\alpha}(\eta)} d\eta}{(n-4)!} + \frac{((\delta'(t))_+)^2}{4\delta(t)}.$$

This implies that

$$\int_{t_1}^s \left[\delta(t) \frac{\int_t^\infty (\eta - t)^{n-4} \frac{[\int_\eta^\infty q(s)ds]^{1/\alpha}}{r^{1/\alpha}(\eta)} d\eta}{(n-4)!} - \frac{((\delta'(t))_+)^2}{4\delta(t)} \right] dt \leq w(t_1)$$

for all large s , which contradicts (2.4). Therefore, every solution of (1.4) is oscillatory. \square

Let $\rho(t) = t^{n-1}$ and $\delta(t) = t$. As a consequence of Theorem 2.3, we obtain the following oscillation criterion.

Corollary 2.4. *Let $n \geq 4$ be even and (1.5) hold. Assume for some constant $\lambda_0 \in (0, 1)$,*

$$\int^\infty \left[t^{n-1} q(t) - \frac{((n-2)!)^\alpha (n-1)^{\alpha+1}}{(\alpha+1)^{\alpha+1} \lambda_0^\alpha} r(t) t^{-n\alpha+n+\alpha-2} \right] dt = \infty \tag{2.12}$$

and, one of conditions (2.2), (2.3), and

$$\int^\infty \left[t \frac{\int_t^\infty (\eta - t)^{n-4} \frac{[\int_\eta^\infty q(s)ds]^{1/\alpha}}{r^{1/\alpha}(\eta)} d\eta}{(n-4)!} - \frac{1}{4t} \right] dt = \infty \tag{2.13}$$

holds. Then (1.4) is oscillatory.

As an application of Corollary 2.4, we give the following.

Example 2.5. Consider the equation

$$x^{(4)}(t) + \frac{a_0}{t^4} x(2t) = 0, \tag{2.14}$$

where $t \geq 1$ and $a_0 > 0$ is a constant. Let $n = 4$, $\alpha = 1$, $r(t) = 1$, $q(t) = a_0/t^4$, and $\tau(t) = 2t$. Then

$$\begin{aligned} & \int^\infty \left[t^{n-1} q(t) - \frac{((n-2)!)^\alpha (n-1)^{\alpha+1}}{(\alpha+1)^{\alpha+1} \lambda_0^\alpha} r(t) t^{-n\alpha+n+\alpha-2} \right] dt \\ &= \left[a_0 - \frac{9}{2\lambda_0} \right] \int^\infty \frac{dt}{t} = \infty, \quad \text{if } a_0 > \frac{9}{2\lambda_0} \quad \text{for some } \lambda_0 \in (0, 1) \end{aligned}$$

and

$$\int^\infty \left[t \frac{\int_t^\infty (\eta - t)^{n-4} \frac{[\int_\eta^\infty q(s)ds]^{1/\alpha}}{r^{1/\alpha}(\eta)} d\eta}{(n-4)!} - \frac{1}{4t} \right] dt = \left[\frac{a_0}{6} - \frac{1}{4} \right] \int^\infty \frac{dt}{t} = \infty, \quad \text{if } a_0 > \frac{3}{2}.$$

Hence by Corollary 2.4, (2.14) is oscillatory if $a_0 > 9/(2\lambda_0)$ for some constant $\lambda_0 \in (0, 1)$. For example, one can take $a_0 > 50/11 \approx 4.55$ (let $\lambda_0 = 99/100$).

In the following, we compare our result with those of [2,5,11]. First,

$$\liminf_{t \rightarrow \infty} \int_t^{\tau(t)} [\tau(s) - s]^{n-2} \left(\int_{\tau(s)}^\infty q(u) du \right)^{1/\alpha} ds = \frac{a_0}{24} \liminf_{t \rightarrow \infty} \int_t^{2t} \frac{ds}{s} = \frac{a_0 \ln 2}{24} > \frac{2}{e}, \quad \text{if } a_0 > \frac{48}{e \ln 2}.$$

Hence by [Theorem 1.1](#), we have that (2.14) is oscillatory if $a_0 > 48/(e \ln 2)$. Therefore, our result improves [Theorem 1.1](#). Next,

$$\limsup_{t \rightarrow \infty} \frac{t^{n-1}}{(n-1)!} \left[\int_t^\infty q(s) ds + \frac{\theta \alpha}{2(n-2)!} \int_t^\infty s^{n-2} \left(\int_s^\infty q(v) dv \right)^{(\alpha+1)/\alpha} ds \right]^{1/\alpha} = \frac{a_0}{18} + \frac{\theta a_0^2}{648}.$$

One can easily see that

$$\frac{a_0}{18} + \frac{\theta a_0^2}{648} < 1, \quad \text{if } a_0 < 18 \text{ for some constant } \theta \in (0, 1).$$

Therefore, our result improves [Theorem 1.2](#). Note that [Theorem 1.3](#) (let $\rho(t) = t^3$) cannot be applied in (2.14) due to the fact that, for all $\theta > \max\{1, a_0/9\}$, the formula

$$[a_0 - 9\theta] \int_t^\infty \frac{dt}{t} = \infty$$

does not hold. Hence our result improves [Theorem 1.3](#). Next, for $\alpha = 1$ and $n = 4$, we have

$$\limsup_{t \rightarrow \infty} t^{\alpha(n-1)} \int_t^\infty q(s) ds = \frac{a_0}{3} > 3!, \quad \text{if } a_0 > 18.$$

Therefore, our result improves [Theorem 1.4](#). Finally, letting $\rho(t) = t^3$ and using [Theorem 1.5](#), we obtain that (2.14) is oscillatory if $a_0 > 1728$. Therefore, our result improves [Theorem 1.5](#).

Next, we consider the case where (1.6) holds.

Theorem 2.6. *Let $n \geq 4$ be even and (1.6) hold. Assume that there exist two functions $\rho, \delta \in C^1([t_0, \infty), (0, \infty))$ such that (2.1) holds for some constant $\lambda_0 \in (0, 1)$ and, one of conditions (2.2), (2.3), and (2.4) is satisfied. If for some constant $\lambda_1 \in (0, 1)$,*

$$\int_t^\infty \left[q(t) \left(\frac{\lambda_1}{(n-2)!} \tau^{n-2}(t) \right)^\alpha R^\alpha(\tau(t)) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{1}{R(t)r^{1/\alpha}(t)} \right] dt = \infty, \tag{2.15}$$

then every solution of (1.4) is either oscillatory or satisfies $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof. Assume that (1.4) has a nonoscillatory solution x . Without loss of generality, we may assume that x is eventually positive. It follows from (1.4) and [Lemma 2.1](#) that there exist three possible cases for $t \geq t_1$ large enough: (1), (2) (as those of the proof of [Theorem 2.3](#)), and

$$(3) \quad x(t) > 0, x^{(n-2)}(t) > 0, x^{(n-1)}(t) < 0, (r(x^{(n-1)})^\alpha)'(t) \leq 0.$$

The proof of the case where (1) or (2) holds is the same as that of [Theorem 2.3](#). Assume now that case (3) holds and $\lim_{t \rightarrow \infty} x(t) \neq 0$. Noting that $r(x^{(n-1)})^\alpha$ is nonincreasing, we have

$$r^{1/\alpha}(s)x^{(n-1)}(s) \leq r^{1/\alpha}(t)x^{(n-1)}(t), \quad s \geq t \geq t_1.$$

Dividing the latter inequality by $r^{1/\alpha}(s)$ and integrating the resulting inequality from t to l , we obtain

$$x^{(n-2)}(l) \leq x^{(n-2)}(t) + r^{1/\alpha}(t)x^{(n-1)}(t) \int_t^l r^{-1/\alpha}(s) ds.$$

Letting $l \rightarrow \infty$, we get

$$0 \leq x^{(n-2)}(t) + r^{1/\alpha}(t)x^{(n-1)}(t)R(t),$$

which yields

$$-\frac{r^{1/\alpha}(t)x^{(n-1)}(t)}{x^{(n-2)}(t)}R(t) \leq 1. \tag{2.16}$$

Furthermore, we have

$$\left(\frac{x^{(n-2)}}{R} \right)'(t) \geq 0 \tag{2.17}$$

due to (2.16). Define the function w by

$$w(t) := \frac{r(t)(x^{(n-1)}(t))^\alpha}{(x^{(n-2)}(t))^\alpha}, \quad t \geq t_1. \tag{2.18}$$

Then $w(t) < 0$ for $t \geq t_1$. Using (2.16) and (2.18), we see that

$$-w(t)R^\alpha(t) \leq 1. \tag{2.19}$$

Differentiating (2.18), we have

$$w'(t) = \frac{(r(t)(x^{(n-1)}(t))^\alpha)'}{(x^{(n-2)}(t))^\alpha} - \alpha \frac{r(t)(x^{(n-1)}(t))^{\alpha+1}}{(x^{(n-2)}(t))^{\alpha+1}}.$$

It follows from (1.4) and (2.18) that

$$w'(t) = -q(t) \frac{x^\alpha(\tau(t))}{(x^{(n-2)}(t))^\alpha} - \alpha \frac{w^{(\alpha+1)/\alpha}(t)}{r^{1/\alpha}(t)}.$$

On the other hand, by Lemma 2.2, we obtain

$$x(t) \geq \frac{\lambda}{(n-2)!} t^{n-2} x^{(n-2)}(t)$$

for every $\lambda \in (0, 1)$ and for all sufficiently large t . Then, we have

$$\begin{aligned} w'(t) &= -q(t) \frac{x^\alpha(\tau(t))}{(x^{(n-2)}(\tau(t)))^\alpha} \frac{(x^{(n-2)}(\tau(t)))^\alpha}{(x^{(n-2)}(t))^\alpha} - \alpha \frac{w^{(\alpha+1)/\alpha}(t)}{r^{1/\alpha}(t)} \\ &\leq -q(t) \left(\frac{R(\tau(t))}{R(t)} \right)^\alpha \left(\frac{\lambda}{(n-2)!} \tau^{n-2}(t) \right)^\alpha - \alpha \frac{w^{(\alpha+1)/\alpha}(t)}{r^{1/\alpha}(t)} \end{aligned} \tag{2.20}$$

due to (2.17). Multiplying (2.20) by $R^\alpha(t)$ and integrating the resulting inequality from t_1 to t , we conclude that

$$\begin{aligned} R^\alpha(t)w(t) - R^\alpha(t_1)w(t_1) + \alpha \int_{t_1}^t r^{-1/\alpha}(s)R^{\alpha-1}(s)w(s)ds + \int_{t_1}^t q(s) \left(\frac{\lambda}{(n-2)!} \tau^{n-2}(s) \right)^\alpha R^\alpha(\tau(s))ds \\ + \alpha \int_{t_1}^t \frac{w^{(\alpha+1)/\alpha}(s)}{r^{1/\alpha}(s)} R^\alpha(s)ds \leq 0. \end{aligned}$$

Set $B := r^{-1/\alpha}(s)R^{\alpha-1}(s)$, $A := R^\alpha(s)/r^{1/\alpha}(s)$, and $v := -w(s)$. Using the inequality

$$Av^{(\alpha+1)/\alpha} - Bv \geq -\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}, \quad A > 0,$$

we have, for every constant $\lambda \in (0, 1)$ and for all large t ,

$$\int_{t_1}^t \left[q(s) \left(\frac{\lambda}{(n-2)!} \tau^{n-2}(s) \right)^\alpha R^\alpha(\tau(s)) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{1}{R(s)r^{1/\alpha}(s)} \right] ds \leq R^\alpha(t_1)w(t_1) + 1$$

due to (2.19), which contradicts (2.15). Therefore, every solution of (1.4) is either oscillatory or satisfies $\lim_{t \rightarrow \infty} x(t) = 0$. \square

As an application of Theorem 2.6, we give the following.

Example 2.7. Consider the equation

$$(e^t x'''(t))' + \frac{e^{t+\frac{1}{2}}}{16} x(t+1) = 0, \tag{2.21}$$

where $t \geq 1$. Let $\alpha = 1$ and $r(t) = e^t$. Then $R(t) = e^{-t}$. It is easy to see that all conditions of Theorem 2.6 are satisfied. Hence every solution of (2.21) is oscillatory or satisfies $\lim_{t \rightarrow \infty} x(t) = 0$. In fact, one such solution is $x(t) = e^{-\frac{t}{2}}$. Note that all results of [23,24] cannot be applied to (2.21), since $\tau(t) = t + 1 > t$.

Finally, we give a new result which differs from Theorem 2.6.

Theorem 2.8. Let $n \geq 4$ be even and (1.6) hold. Assume that there exist two functions $\rho, \delta \in C^1([t_0, \infty), (0, \infty))$ such that (2.1) holds for some constant $\lambda_0 \in (0, 1)$ and, one of conditions (2.2), (2.3), and (2.4) is satisfied. Suppose further that (2.15) holds for some constant $\lambda_1 \in (0, 1)$. If there exists a function $g \in C([t_0, \infty), \mathbb{R})$ such that $g(t) \geq \tau(t)$, $g(t) > t$ for $t \geq t_0$, and

$$\liminf_{t \rightarrow \infty} \int_t^{g(t)} q(s)(V(g(s)))^\alpha ds > \frac{1}{e}, \tag{2.22}$$

where

$$V(t) := \frac{\int_t^\infty (\eta - t)^{n-3} R(\eta) d\eta}{(n-3)!}, \tag{2.23}$$

then (1.4) is oscillatory.

Proof. Assume that (1.4) has a nonoscillatory solution x . Without loss of generality, we may assume that x is eventually positive. It follows from (1.4) and Lemma 2.1 that there exist four possible cases for $t \geq t_1$ large enough: (1), (2) (as those of the proof of

Theorem 2.3), and

- (3) $x(t) > 0, x'(t) > 0, x^{(n-2)}(t) > 0, x^{(n-1)}(t) \leq 0, (r(x^{(n-1)})^\alpha)'(t) \leq 0;$
- (4) $x(t) > 0, x^{(j)}(t) < 0, x^{(j+1)}(t) > 0$ for every odd integer $j \in \{1, 2, \dots, n-3\}, x^{(n-1)}(t) < 0,$ and $(r(x^{(n-1)})^\alpha)'(t) \leq 0.$

The proof of the case where (1) or (2) holds is the same as that of Theorem 2.3. Note that when (3) holds, we have that $\lim_{t \rightarrow \infty} x(t) \neq 0.$ Proceeding as in the proof of Theorem 2.6, we can get a contradiction to (2.15). Assume now that case (4) holds. From the proof of Theorem 2.6, we have (2.16). That is,

$$x^{(n-2)}(t) \geq -R(t)r^{1/\alpha}(t)x^{(n-1)}(t). \tag{2.24}$$

Integrating (2.24) from t to $\infty,$ we obtain

$$-x^{(n-3)}(t) \geq -r^{1/\alpha}(t)x^{(n-1)}(t) \int_t^\infty R(s)ds.$$

Similarly, integrating the latter inequality from t to ∞ for a total of $(n-3)$ times, we find that

$$x(t) \geq \frac{-r^{1/\alpha}(t)x^{(n-1)}(t)}{(n-3)!} \int_t^\infty (\eta-t)^{n-3}R(\eta)d\eta. \tag{2.25}$$

It follows from $x' < 0$ and (1.4) that

$$(r(t)(x^{(n-1)}(t))^\alpha)' + q(t)x^\alpha(g(t)) \leq 0.$$

Hence by (2.25), we have

$$(r(t)(x^{(n-1)}(t))^\alpha)' - q(t)r(g(t))(x^{(n-1)}(g(t)))^\alpha (V(g(t)))^\alpha \leq 0.$$

That is,

$$(-r(t)(x^{(n-1)}(t))^\alpha)' - q(t)(-r(g(t))(x^{(n-1)}(g(t)))^\alpha)(V(g(t)))^\alpha \geq 0.$$

Thus, $z := -r(x^{(n-1)})^\alpha > 0$ is a positive solution of the advanced differential inequality

$$z'(t) - q(t)(V(g(t)))^\alpha z(g(t)) \geq 0. \tag{2.26}$$

However, using (2.22) and [15, Theorem 2.4.1], we see that (2.26) has no positive solutions. This is a contradiction. Therefore, (1.4) is oscillatory. □

3. Further results

It is well known (see [6]) that the differential equation

$$(a(t)(x'(t))^\alpha)' + q(t)x^\alpha(t) = 0, \tag{3.1}$$

where $\alpha > 0$ is a ratio of odd positive integers, $a \in C([t_0, \infty), \mathbb{R}^+), q \in C([t_0, \infty), \mathbb{R})$ is nonoscillatory if and only if there exist a number $T \geq t_0$ and a function $v \in C^1([t_0, \infty), \mathbb{R})$ which satisfies the inequality

$$v'(t) + \alpha a^{-1/\alpha}(t)(v(t))^{(1+\alpha)/\alpha} + q(t) \leq 0, \quad \text{on } [T, \infty).$$

In the following, we compare oscillatory and asymptotic behavior of Eq. (1.4) with second-order equations of type (3.1). There are many oscillation results for Eq. (3.1); see, e.g., [3,4,17,20].

Theorem 3.1. *Let $n \geq 4$ be even and (1.5) hold. Assume that the equation*

$$\left(\frac{((n-2)!)^\alpha r(t)}{(\lambda_0 t^{n-2})^\alpha} (x'(t))^\alpha \right)' + q(t)x^\alpha(t) = 0 \tag{3.2}$$

is oscillatory for some constant $\lambda_0 \in (0, 1).$ If, in addition, either (2.2) or (2.3) holds, or the equation

$$x''(t) + \frac{\int_t^\infty (\eta-t)^{n-4} \left[\frac{\int_\eta^\infty q(s)ds \right]^{1/\alpha}}{r^{1/\alpha}(\eta)} d\eta}{(n-4)!} x(t) = 0 \tag{3.3}$$

is oscillatory, then (1.4) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.3, we have (2.7) and (2.11). Letting $\rho(t) = 1$ in (2.7), we have

$$u'(t) + \frac{\lambda \alpha t^{n-2}}{(n-2)! r^{1/\alpha}(t)} u^{(\alpha+1)/\alpha}(t) + q(t) \leq 0$$

for every constant $\lambda \in (0, 1)$. Then, we see that Eq. (3.2) is nonoscillatory for every constant $\lambda_0 \in (0, 1)$, which is a contradiction. Letting $\delta(t) = 1$ in (2.11), we have

$$w'(t) + w^2(t) + \frac{\int_t^\infty (\eta - t)^{n-4} \left[\int_\eta^\infty \frac{q(s) ds}{r^{1/\alpha}(\eta)} \right]^{1/\alpha} d\eta}{(n-4)!} \leq 0,$$

which implies that Eq. (3.3) is nonoscillatory. This is a contradiction. The proof is complete. \square

On the basis of Theorem 3.1, we have the following result.

Theorem 3.2. Let $n \geq 4$ be even and (1.6) hold. Assume that Eq. (3.2) is oscillatory for some constant $\lambda_0 \in (0, 1)$. Suppose further that either (2.2) or (2.3) holds, or Eq. (3.3) is oscillatory. If the equation

$$(r(t)(x'(t))^\alpha)' + q(t) \left(\frac{R(\tau(t))}{R(t)} \right)^\alpha \left(\frac{\lambda_1}{(n-2)!} \tau^{n-2}(t) \right)^\alpha x^\alpha(t) = 0 \quad (3.4)$$

is oscillatory for some constant $\lambda_1 \in (0, 1)$, where R is as in Section 2, then every solution of (1.4) is either oscillatory or satisfies $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof. Proceeding as in the proof of Theorem 2.6, we have (2.20). From (2.20), we have

$$w'(t) + \alpha \frac{w^{(\alpha+1)/\alpha}(t)}{r^{1/\alpha}(t)} + q(t) \left(\frac{R(\tau(t))}{R(t)} \right)^\alpha \left(\frac{\lambda}{(n-2)!} \tau^{n-2}(t) \right)^\alpha \leq 0$$

for every constant $\lambda \in (0, 1)$. Then, we see that Eq. (3.4) is nonoscillatory for every constant $\lambda_1 \in (0, 1)$, which is a contradiction. The proof is complete. \square

On the basis of Theorem 3.2, we have the following result.

Theorem 3.3. Let all conditions of Theorem 3.2 hold. If the equation

$$z'(t) - q(t)(V(g(t)))^\alpha z(g(t)) = 0 \quad (3.5)$$

is oscillatory, where g and V are as in Theorem 2.8, then (1.4) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.8, we have that the advanced differential inequality (2.26) has an eventually positive solution. By virtue of a well known result (see [12]), the corresponding differential Eq. (3.5) also has an eventually positive solution. This contradiction completes the proof. \square

4. Discussion

This paper presents new criteria for oscillation and asymptotic behavior of an even-order advanced differential Eq. (1.4) with power type nonlinearities. With the help of the original results due to Philos [19], new theorems are based on a thorough analysis of possible behavior of nonoscillatory solutions of (1.4); they complement and improve a number of results reported in the cited papers.

Most oscillation results reported in the literature for (1.4) and its particular cases have been obtained under the assumption (1.5) which significantly simplifies the analysis of the behavior of nonoscillatory solutions of (1.4). The study of oscillatory properties of (1.4) in the case (1.5) is more difficult in comparison with second-order differential equations. Since the sign of the derivative x'' is not known, Theorem 2.3 includes a pair of assumptions; see, e.g., (2.1) and (2.4). It follows from Example 2.5 that condition (2.4) plays an auxiliary role in certain sense. That is, condition (2.1) may have more important role in Theorem 2.3.

On the basis of condition (1.6), the study of oscillation and asymptotic behavior of (1.4) brings additional difficulties. In particular, in order to deal with the case when $x^{(n-1)} < 0$ (which is simply eliminated if condition (1.5) holds), we have to impose additional assumptions; see, e.g., (2.15) and (2.22).

In Section 3, the oscillatory and asymptotic properties of (1.4) are detected from oscillation of second-order Eqs. (3.2)–(3.5). As fairly noticed by the referee, the results formulated in Section 2 can be obtained from known oscillation criteria for the second-order Eqs. (3.2)–(3.5). Several explicit criteria (Theorem 2.3, Corollary 2.4, Theorems 2.6 and 2.8) are presented in order to explain advantages of new results.

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