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Existence results for dynamic Sturm–Liouville boundary value problems via variational methods

David Barilla[a], Martin Bohner[b], Shapour Heidarkhani[c,*], Shahin Moradi[c]

[a] Department of Economics, University of Messina, Messina 98122, Italy
[b] Department of Mathematics and Statistics, Missouri S&T, Rolla, MO 65409-0020, USA
[c] Department of Mathematics, Faculty of Sciences, Razi University, Kermanshah 67149, Iran

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ABSTRACT

Several conditions ensuring existence of solutions of a dynamic Sturm–Liouville boundary value problem are derived. Variational methods are utilized in the proofs. An example illustrating the main results is given.

1. Introduction

Let be given any time scale $\mathbb{T}$, namely, a closed and nonempty subset of the real numbers. In particular, $\mathbb{T}=\mathbb{Z}$ (all integers) and $\mathbb{T}=\mathbb{R}$ (all real numbers) are two examples of time scales, and so-called dynamic equations on these time scales correspond to difference equations and differential equations, respectively. Let $T>0$ be fixed and suppose $0, T\in\mathbb{T}$. In this contribution, we study the second-order dynamic Sturm–Liouville BVP (boundary value problem)

$$\begin{cases} -(px^{\Delta})^{\Delta}(t)+q(t)x^{\sigma}(t)=f(t,x^{\sigma}(t)), \quad t\in[0,T]_{\mathbb{T}}, \\ \alpha_1 x(0)-\alpha_2 x^{\Delta}(0)=0, \quad \alpha_3 x(\sigma^2(T))+\alpha_4 x^{\Delta}(\sigma(T))=0, \end{cases} \tag{P^f}$$

where

$$p\in C^1([0,\sigma(T)]_{\mathbb{T}},(0,\infty)), \quad q\in C([0,T]_{\mathbb{T}},[0,\infty)), \quad f\in C([0,T]_{\mathbb{T}}\times\mathbb{R},\mathbb{R}),$$
$$\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_1+\alpha_2\geq 0, \quad \alpha_1+\alpha_3,\alpha_3+\alpha_4>0.$$

During the last thirty years, time scales theory, which was created by Stefan Hilger [23], unifying discrete and continuous analysis, has received considerable attention due to its potential applications in the study of epidemic models, population models, finance, stock market, economics [7], and heat transfer, and it has since then been further developed by numerous authors in several fields of study, see, for example, [18,24,25].

* Corresponding author.
E-mail addresses: dbarilla@unime.it (D. Barilla), bohner@mst.edu (M. Bohner), sh.heidarkhani@razi.ac.ir (S. Heidarkhani), shahin.moradi86@yahoo.com (S. Moradi).

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Recently, BVPs for dynamic equations on time scales have been extensively studied by many researchers. Various methods and techniques have been applied, such as methods of lower and upper solutions, cone-theoretic fixed point theorems, variational methods, and coincidence degree theory. Many existence results for solutions and also for positive solutions have been established, and we refer to [3,5,6,8,11–15,17,20,26,29–32,34] and the references given there. For example, Sun and Li, in [30], by using fixed point theorems in cones, have obtained some new and general results for the existence of single and also of multiple positive solutions for

$$\begin{cases} -x^{\Delta\nabla}(t) + a(t)f(t, x(t)) = 0, & t \in [0, T]_{\mathbb{T}}, \\ \beta x(0) - \gamma x^{\Delta}(0) = 0, & \alpha x(\eta) = x(T) = 0, \end{cases}$$

where

$$\beta, \gamma \geq 0,\ \beta + \gamma > 0,\ \eta \in (0, \rho(T))_{\mathbb{T}},\ 0 < \alpha < \frac{T}{\eta},\ d = \beta(T - \alpha\eta) + \gamma(1 - \alpha) > 0.$$

In [5,6], Agarwal et al. have examined the dynamic equations

$$-x^{\Delta\Delta}(t) = f(t, x^{\sigma}(t)), \quad t \in (a, b)_{\mathbb{T}}$$

and

$$-x^{\Delta\Delta}(t) = f(\sigma(t), x^{\sigma}(t)), \quad t \in (a, b)_{\mathbb{T}}$$

with Dirichlet boundary conditions, and they have obtained certain existence criteria for single and multiple positive solutions by utilizing variational techniques. In [26], Ozkan considered a boundary-value problem involving a dynamic Sturm–Liouville equation and boundary conditions that depend on a spectral parameter, and he also introduced an operator formulation of this problem and gave some properties of its eigenfunctions and eigenvalues. Finally, he determined the number of eigenvalues for a finite time scale.

In [8], under some algebraic assumptions on the nonlinearity, the existence of *at least three* distinct nonnegative solutions of a *double* eigenvalue second-order dynamic Sturm–Liouville boundary value problem has been discussed by utilizing a critical point result. The exact collocation of the eigenvalues has been given. Demanding an additional asymptotical behaviour of the data at zero, nontriviality of the solution has been achieved also, under appropriate assumptions. Here, in the present paper, we discuss existence of *at least one* nontrivial solution of the *single* eigenvalue second-order dynamic Sturm–Liouville boundary value problem (P^f), under a certain asymptotical assumption of the nonlinearity at zero (Theorem 2). In Theorem 3, we present an application of Theorem 2. Moreover, we give some observations and remarks on our results. As a special case of our result, we present Theorem 4, in the case when the function f does not depend on time. Finally, we offer Example 1, in which all hypotheses of Theorem 4 are satisfied.

2. Preliminaries

In this paper, we prove the existence of at least one nontrivial solution for (P^f). The main argument in our results is a famous variational principle by Ricceri [28, Theorem 2.1], in the special form given by Bonanno and Molica Bisci in [16]. This principle has been extensively applied to a variety of problems, and we refer to [1,2,9,10,19,21,22].

Theorem 1. *Assume $\mathcal{B}$ is a real and reflexive Banach space. Let be given two Gâteaux-differentiable functionals $\mathcal{J}_1, \mathcal{J}_2 : \mathcal{B} \to \mathbb{R}$ so that $\mathcal{J}_1$ is strongly continuous, sequentially weakly lower semicontinuous, and coercive in $\mathcal{B}$, and $\mathcal{J}_2$ is sequentially weakly upper semicontinuous in $\mathcal{B}$. Define the functional I_ν by $I_\nu := \mathcal{J}_1 - \nu\mathcal{J}_2,\ \nu \in \mathbb{R}$. Moreover, for any $s > \inf_{\mathcal{B}} \mathcal{J}_1$, define the function φ by*

$$\varphi(s) := \inf_{x \in \mathcal{J}_1^{-1}(-\infty, s)} \frac{\sup_{y \in \mathcal{J}_1^{-1}(-\infty, s)} \mathcal{J}_2(y) - \mathcal{J}_2(x)}{s - \mathcal{J}_1(x)}.$$

Then, for any $s > \inf_{\mathcal{B}} \mathcal{J}_1$ and any $\nu \in \left(0, \frac{1}{\varphi(s)}\right)$, the restriction of the functional I_ν to $\mathcal{J}_1^{-1}(-\infty, s)$ has a global minimum, which is a critical point (more precisely, a local minimum) of I_ν in $\mathcal{B}$.

Now we introduce some basic definitions and auxiliary results that are used in this paper. We define (see [4, Definition 2.4])

$$L^1_{\Delta}([t_1, t_2)_{\mathbb{T}}) := \left\{ f : [t_1, t_2)_{\mathbb{T}} \to \mathbb{R} : \int_{[t_1, t_2)_{\mathbb{T}}} |f(s)| \Delta s < \infty \right\}$$

and

$$L^2_{\Delta}([t_1, t_2)_{\mathbb{T}}) := \left\{ f : [t_1, t_2)_{\mathbb{T}} \to \mathbb{R} : \int_{[t_1, t_2)_{\mathbb{T}}} |f(s)|^2 \Delta s < \infty \right\}.$$

For $f \in L^1_{\Delta}([t_1, t_2)_{\mathbb{T}})$, we introduce for convenience the notation

$$\int_{t_1}^{t_2} f(s) \Delta s = \int_{[t_1, t_2)_{\mathbb{T}}} f(s) \Delta s.$$

When studying (P^f), the space in our variational setting is given by

$$\begin{aligned}\mathcal{H} &:= \mathrm{H}^1_\Delta([0,\sigma^2(T)]_{\mathbb{T}})\\ &:= \left\{x : [0,\sigma^2(T)]_{\mathbb{T}} \to \mathbb{R} :\ x \in \mathrm{AC}[0,\sigma^2(T)]_{\mathbb{T}} \text{ and } x^\Delta \in \mathrm{L}^2_\Delta([0,\sigma^2(T))_{\mathbb{T}})\right\}.\end{aligned}$$

Then $\mathcal{H}$ is known (see [33]) to be a Hilbert space when equipped with the inner product

$$(x,y)_{\mathcal{H}} = \int_0^{\sigma^2(T)} x(t)y(t)\Delta t + \int_0^{\sigma^2(T)} x^\Delta(t)y^\Delta(t)\Delta t,$$

and we use the notation $\|\cdot\|_{\mathcal{H}}$ to denote the norm that is induced by $(\cdot,\cdot)_{\mathcal{H}}$. For $x, y \in \mathcal{H}$, we introduce

$$(x,y)_0 = \int_0^{\sigma^2(T)} p(t)x^\Delta(t)y^\Delta(t)\Delta t + \int_0^{\sigma(T)} q(t)x^\sigma(t)y^\sigma(t)\Delta t + \beta_1 p(0)x(0)y(0) + \beta_2 p(\sigma(T))x(\sigma^2(T))y(\sigma^2(T)),$$

where

$$\beta_1 = \begin{cases}\dfrac{\alpha_1}{\alpha_2} & \text{if } \alpha_2 > 0,\\ 0 & \text{if } \alpha_2 = 0\end{cases} \tag{1}$$

and

$$\beta_2 = \begin{cases}\dfrac{\alpha_3}{\alpha_4} & \text{if } \alpha_4 > 0,\\ 0 & \text{if } \alpha_4 = 0.\end{cases} \tag{2}$$

We let $\|x\|_0$ be the norm that is induced by $(x, y)_0$.

Lemma 1 (See [32, Lemmas 2.1, 2.2 and 4.2]). *The immersion $\mathcal{H} \hookrightarrow \mathrm{C}([0,\sigma^2(T)]_{\mathbb{T}})$ is compact. If $x \in \mathcal{H}$, then*

$$|x(t)| \le \sqrt{2}\max\{(\sigma^2(T))^{\frac{1}{2}}, (\sigma^2(T))^{-\frac{1}{2}}\}\|x\|_{\mathcal{H}} \text{ for every } t \in [0,\sigma^2(T)]_{\mathbb{T}}.$$

If $\alpha_2, \alpha_4 > 0$ or $q(t) > 0$ for $t \in [0,T]_{\mathbb{T}}$, then for $x \in \mathcal{H}$, $|x(t)| \le C\|x\|_0$ for every $t \in [0,\sigma^2(T)]_{\mathbb{T}}$, where $C = \min\{M_1, M_2, M_3\}$ and

$$M_1 = \sqrt{2}\max\left\{\frac{1}{\sqrt{\beta_1 p(0)}}, \frac{\sqrt{\sigma^2(T)}}{\underline{p}}\right\},$$

$$M_2 = \sqrt{2}\max\left\{\frac{1}{\sqrt{\beta_2 p(0)}}, \frac{\sqrt{\sigma^2(T)}}{\underline{p}}\right\},$$

$$M_3 = \sqrt{2}\max\left\{\frac{\sqrt{\sigma(T)}}{\underline{q}}, \frac{\sqrt{\sigma^2(T)}}{\underline{p}}\right\},$$

and where

$$\underline{p} = \min_{t\in[0,\sigma(T)]_{\mathbb{T}}} p(t) \quad \text{and} \quad \underline{q} = \min_{t\in[0,T]_{\mathbb{T}}} q(t).$$

3. Main results

We now give our main result for (P^f).

Theorem 2. *Assume*

$$\sup_{\theta>0}\frac{\theta^2}{F_\theta} > 2C^2, \quad \text{where} \quad F_\theta = \int_0^{\sigma(T)} \max_{|\xi|\le\theta} F(t,\xi)\Delta t. \tag{S}$$

Then, (P^f) has at least one solution in $\mathcal{H}$.

Proof. We will apply Theorem 1 to (P^f). Let $\mathcal{B} = \mathcal{H}$. Let us introduce the two functionals $\mathcal{J}_1, \mathcal{J}_2$ by

$$\mathcal{J}_1(x) = \frac{1}{2}\|x\|_0^2 \tag{3}$$

and

$$\mathcal{J}_2(x) = \int_0^{\sigma(T)} F(t, x^\sigma(t))\Delta t \tag{4}$$

for $x \in \mathcal{B}$. We define

$$I(x) = \mathcal{J}_1(x) - \mathcal{J}_2(x) \quad \text{for} \quad x \in \mathcal{B}.$$

We now show that $\mathcal{J}_1$ and $\mathcal{J}_2$ fulfill the conditions assumed in Theorem 1. As $\mathcal{B}$ is compactly embedded in $(C^0([0,T]_{\mathbb{T}}), \mathbb{R})$, it is easy to see that $\mathcal{J}_2$ is Gâteaux differentiable, and the Gâteaux derivative of $\mathcal{J}_2$ at $x \in \mathcal{B}$ is $\mathcal{J}_2'(x) \in \mathcal{B}^*$ given as

$$\mathcal{J}_2'(x)(y) = \int_0^{\sigma(T)} f(t, x^\sigma(t)) y^\sigma(t) \Delta t \quad \text{for all} \quad y \in \mathcal{B},$$

and $\mathcal{J}_2$ is sequentially weakly upper semicontinuous. Additionally, $\mathcal{J}_1$ is also Gâteaux differentiable, and the Gâteaux derivative of $\mathcal{J}_1$ at $x \in \mathcal{B}$ is $\mathcal{J}_1'(x) \in \mathcal{B}^*$ given as

$$\begin{aligned}
\mathcal{J}_1'(x)(y) &= \frac{\mathrm{d}}{\mathrm{d}\nu} \mathcal{J}_1(x + \nu y)\bigg|_{\nu=0} = \frac{\mathrm{d}}{2\mathrm{d}\nu} \|x + \nu y\|_0^2 \bigg|_{\nu=0} \\
&= \frac{\mathrm{d}}{2\mathrm{d}\nu} \bigg\{ \int_0^{\sigma^2(T)} p(t)(x^\Delta(t) + \nu y^\Delta(t))^2 \Delta t + \int_0^{\sigma(T)} q(t)(x^\sigma(t) + \nu y^\sigma(t))^2 \Delta t \\
&\quad + \beta_1 p(0)(x(0) + \nu y(0))^2 + \beta_2 p(\sigma(T))(x(\sigma^2(T)) + \nu y(\sigma^2(T)))^2 \bigg\} \bigg|_{\nu=0} \\
&= \int_0^{\sigma^2(T)} p(t)(x^\Delta(t) + \nu y^\Delta(t)) y^\Delta(t) \Delta t + \int_0^{\sigma(T)} q(t)(x^\sigma(t) + \nu y^\sigma(t)) y^\sigma(t) \Delta t \\
&\quad + \beta_1 p(0)(x(0) + \nu y(0)) y(0) + \beta_2 p(\sigma(T))(x(\sigma^2(T)) + \nu y(\sigma^2(T))) x(\sigma^2(T)) \bigg|_{\nu=0} \\
&= \int_0^{\sigma^2(T)} p(t) x^\Delta(t) y^\Delta(t) \Delta t + \int_0^{\sigma(T)} q(t) x^\sigma(t) y^\sigma(t) \Delta t \\
&\quad + \beta_1 p(0) x(0) y(0) + \beta_2 p(\sigma(T)) x(\sigma^2(T)) y(\sigma^2(T))
\end{aligned}$$

for all $y \in \mathcal{B}$. Furthermore, we see that $\mathcal{J}_1$ is coercive and sequentially weakly lower semicontinuous. From (S), we can find $\bar{\theta} > 0$ satisfying

$$\frac{\bar{\theta}^2}{F_{\bar{\theta}}} > 2C^2. \tag{5}$$

Define

$$s = \frac{\bar{\theta}^2}{2C^2}.$$

If $x \in \mathcal{J}_1^{-1}(-\infty, s)$, then $\mathcal{J}_1(x) < s$, that is, $\frac{1}{2}\|x\|_0^2 < s$. Hence, by Lemma 1, we obtain $|x(t)| \le C\sqrt{2s} = \bar{\theta}$ for all $t \in [0, \sigma^2(T)]_{\mathbb{T}}$. So

$$\sup_{\mathcal{J}_1(x) < s} \mathcal{J}_2(x) \le F_{\bar{\theta}}.$$

From the above, since $0 \in \mathcal{J}_1^{-1}(-\infty, s)$ and $\mathcal{J}_1(0) = \mathcal{J}_2(0) = 0$, we get

$$\begin{aligned}
\varphi(s) &= \inf_{x \in \mathcal{J}_1^{-1}(-\infty, s)} \frac{(\sup_{y \in \mathcal{J}_1^{-1}(-\infty, s)} \mathcal{J}_2(y)) - \mathcal{J}_2(x)}{s - \mathcal{J}_1(x)} \\
&\le \frac{\sup_{x \in \mathcal{J}_1^{-1}(-\infty, s)} \mathcal{J}_2(x)}{s} \\
&\le 2C^2 \frac{F_{\bar{\theta}}}{\bar{\theta}^2}.
\end{aligned}$$

Thus, it follows that

$$\varphi(s) \le 2C^2 \frac{F_{\bar{\theta}}}{\bar{\theta}^2}. \tag{6}$$

Therefore, by (5) and (6), we get $\varphi(s) < 1$. Hence, as $1 \in \left(0, \frac{1}{\varphi(s)}\right)$, Theorem 1 implies that I has at least one critical point (more precisely, local minimum) $\tilde{x} \in \mathcal{J}_1^{-1}(-\infty, s)$. Thus, using that the solutions of (P^f) are exactly the critical points of I, we obtain the claim. □

We remark that Theorem 2 can also be used to ensure the existence of a solution for the parametric problem

$$\begin{cases} -(px^{\Delta})^{\Delta}(t) + q(t)x^{\sigma}(t) = \nu f(t, x^{\sigma}(t)), \quad t \in [0, T]_{\mathbb{T}}, \\ \alpha_1 x(0) - \alpha_2 x^{\Delta}(0) = 0, \quad \alpha_3 x(\sigma^2(T)) + \alpha_4 x^{\Delta}(\sigma(T)) = 0, \end{cases} \tag{P_ν^f}$$

where $\nu > 0$ is a parameter. This result about (P_ν^f) is given as follows.

Theorem 3. *For all*

$$\nu \in \left(0, \frac{1}{2C^2} \sup_{\theta > 0} \frac{\theta^2}{F_\theta}\right),$$

(P_ν^f) *admits a solution* $x_\nu \in \mathcal{H}$.

Proof. Let ν be in the stated interval. Suppose $\mathcal{J}_1$ and $\mathcal{J}_2$ are as in (3) and (4). Define

$$I_\nu(x) = \mathcal{J}_1(x) - \nu \mathcal{J}_2(x) \quad \text{for all} \quad x \in \mathcal{H}.$$

Thus, we obtain the existence of $\bar{\theta} > 0$ satisfying

$$2C^2 \nu < \frac{\bar{\theta}^2}{F_{\bar{\theta}}}.$$

Put

$$s = \frac{\bar{\theta}^2}{2C^2}.$$

Using the notations from Theorem 2, we get

$$\varphi(s) \le 2C^2 \frac{F_{\bar{\theta}}}{\bar{\theta}^2} < \frac{1}{\nu}.$$

Then, as $\nu \in \left(0, \frac{1}{\varphi(s)}\right)$, Theorem 1 implies that I_ν has at least one critical point (more precisely, local minimum) x_ν in $\mathcal{J}_1^{-1}(-\infty, s)$, and recalling that critical points of I_ν are solutions of (P_ν^f), we have the conclusion. □

4. Remarks, applications, example

We give some implications of the above results.

Remark 1. We remark that, in general, I_ν may be unbounded in $\mathcal{H}$. Indeed, e.g., when $f(\xi) = 1 + |\xi|^{\gamma-2}\xi$ for $\xi \in \mathbb{R}$ with $\gamma > 2$, for any $x \in \mathcal{H} \setminus \{0\}$ and $\upsilon \in \mathbb{R}$, we get

$$I_\nu(\upsilon x) = \mathcal{J}_1(\upsilon x) - \nu \int_0^{\sigma(T)} F(\upsilon x(t))\Delta t \le \frac{\upsilon^2}{2}\|x\|_0^2 - \nu\upsilon|\sigma(T)| - \nu\frac{\upsilon^\gamma}{\gamma}|\sigma(T)|^\gamma \to -\infty$$

as $\upsilon \to \infty$. Therefore, the condition [27, (I_2), Theorem 2.2] is not fulfilled. Hence, we cannot use the direct minimization approach to find the critical points of I_ν.

Now we show that I_ν associated with (P_ν^f) is, in general, not coercive. For example, when $F(\xi) = |\xi|^s$ for $\xi \in \mathbb{R}$ with $s \in (2, \infty)$, for any $x \in \mathcal{H} \setminus \{0\}$ and $\upsilon \in \mathbb{R}$, we get

$$\begin{aligned} I_\nu(\upsilon x) &= \mathcal{J}_1(\upsilon x) - \nu \int_0^{\sigma(T)} F(\upsilon x(t))\Delta t \\ &\le \frac{\upsilon^2}{2}\|x\|_0^2 - \nu\upsilon^s|\sigma(T)|^s \to -\infty \end{aligned}$$

as $\upsilon \to -\infty$.

Remark 2. If in Theorem 2, $f(t, \xi) \ge 0$ for almost every $(t, \xi) \in [0, T]_{\mathbb{T}} \times \mathbb{R}$, then (S) becomes the simpler form

$$\sup_{\theta > 0} \frac{\theta^2}{\int_0^{\sigma(T)} F(t, \theta)\Delta t} > 2C^2. \tag{S_ν}$$

Additionally, if

$$\limsup_{\theta \to \infty} \frac{\theta^2}{\int_0^{\sigma(T)} F(t, \theta)\Delta t} > 2C^2,$$

then (S_ν) automatically holds.

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Remark 3. Assume $\bar{\theta} > 0$ is fixed and

$$\frac{\bar{\theta}^2}{F_{\bar{\theta}}} > 2C^2.$$

Then the result of Theorem 3 holds with $\|x_\nu\|_0 \leq \bar{\theta}$.

Remark 4. If, in Theorem 3, $f(t, 0) \neq 0$ for almost every $t \in [0, T]_{\mathbb{T}}$, then the solution that is obtained is obviously nontrivial. But the nontriviality of this solution can also be verified when $f(t, 0) = 0$ for almost every $t \in [0, T]_{\mathbb{T}}$, requiring the following additional condition at zero: There exist an open set $\emptyset \neq D \subseteq (0, T)_{\mathbb{T}}$ and $B \subset D$ with positive Lebesgue measure such that

$$\limsup_{\xi \to 0^+} \frac{\text{ess inf}_{t\in B} F(t, \xi)}{|\xi|^2} = \infty \quad \text{and} \quad \liminf_{\xi \to 0^+} \frac{\text{ess inf}_{t\in D} F(t, \xi)}{|\xi|^2} > -\infty.$$

To see this, let $0 < \bar{\nu} < \nu^*$, where

$$\nu^* = \frac{1}{2C^2} \sup_{\theta > 0} \frac{\theta^2}{F_\theta}.$$

Then, we obtain the existence of $\bar{\theta} > 0$ satisfying

$$2C^2 \bar{\nu} < \frac{\bar{\theta}^2}{F_{\bar{\theta}}}.$$

According to Theorem 1, for each $\nu \in (0, \bar{\nu})$, I_ν possesses a critical point $x_\nu \in \mathcal{J}_1^{-1}(-\infty, s_\nu)$, where $s_\nu = \frac{\bar{\theta}^2}{2C^2}$. In particular, x_ν is a global minimum of the restriction of I_ν to $\mathcal{J}_1^{-1}(-\infty, s_\nu)$. We proceed to show that x_ν is nontrivial. To do so, we prove

$$\limsup_{\|x\| \to 0^+} \frac{\mathcal{J}_2(x)}{\mathcal{J}_1(x)} = \infty. \tag{7}$$

According to our assumptions at zero, we can find $\zeta > 0$ and κ and a sequence $\{\xi_n\} \subset \mathbb{R}^+$ converging to zero, satisfying

$$\lim_{n\to\infty} \frac{\text{ess inf}_{t\in B} F(t, \xi_n)}{|\xi_n|^2} = \infty$$

and

$$\text{ess inf}_{t\in D} F(t, \xi) \geq \kappa |\xi|^2 \quad \text{for all} \quad \xi \in [0, \zeta].$$

Now, take $C \subset B$ of positive measure and $y \in \mathcal{H}$ with

(i) $y(t) \in [0, 1]$ for all $t \in [0, T]_{\mathbb{T}}$,
(ii) $y(t) = 1 \in \mathbb{R}$ for all $t \in C$,
(iii) $y(t) = 0$ for all $t \in (0, T)_{\mathbb{T}} \setminus D$.

Take $Y > 0$ and let $\eta > 0$ be such that

$$Y < \frac{\eta \text{meas}(C) + \kappa \int_{D\setminus C} |y(t)|^2 \Delta t}{\frac{1}{2}\|y\|_0^2}.$$

Then, there exists $n_0 \in \mathbb{N}$ with $\xi_n < \zeta$ and

$$\text{ess inf}_{t\in B} F(t, \xi_n) \geq \eta |\xi_n|^2$$

for all $n > n_0$. Next, for all $n > n_0$, by using the properties of y (i.e., $0 \leq \xi_n y(t) < \zeta$ for large enough n), we get

$$\begin{aligned} \frac{\mathcal{J}_2(\xi_n y)}{\mathcal{J}_1(\xi_n y)} &= \frac{\int_C F(t, \xi_n) \Delta t + \int_{D\setminus C} F(t, \xi_n y(t)) \Delta t}{\mathcal{J}_1(\xi_n y)} \\ &> \frac{\eta \text{meas}(C) + \kappa \int_{D\setminus C} |y(t)|^2 \Delta t}{\frac{1}{2}\|y\|_0^2} > Y. \end{aligned}$$

As Y is arbitrary, we obtain

$$\lim_{n\to\infty} \frac{\mathcal{J}_2(\xi_n y)}{\mathcal{J}_1(\xi_n y)} = \infty,$$

from which (7) follows. Thus, there exists $\{w_n\} \subset \mathcal{H}$ that converges strongly to zero, $w_n \in \mathcal{J}_1^{-1}(-\infty, s)$, and

$$I_\nu(w_n) = \mathcal{J}_1(w_n) - \nu \mathcal{J}_2(w_n) < 0.$$

As x_ν is a global minimum of the restriction of I_ν to $\mathcal{J}_1^{-1}(-\infty, s)$, we conclude

$$I_\nu(x_\nu) < 0, \tag{8}$$

and thus x_ν is nontrivial.

From (8), we also get that the map

$$(0,\nu^*)\ni\nu\mapsto I_\nu(x_\nu) \tag{9}$$

is negative. Further, we have

$$\lim_{\nu\to 0^+}\|x_\nu\|_0=0.$$

In fact, by noting that $\mathcal{J}_1$ is coercive and that for $\nu\in(0,\nu^*)$, $x_\nu\in\mathcal{J}_1^{-1}(-\infty,s)$, one obtains existence of a constant $\mathcal{L}>0$ with $\|x_\nu\|\le\mathcal{L}$ for all $\nu\in(0,\nu^*)$. This implies existence of $\mathcal{M}>0$ satisfying

$$\left|\int_0^{\sigma(T)} f(t,x_\nu^\sigma(t))x_\nu^\sigma(t)\Delta t\right|\le\mathcal{M}\|x_\nu\|_0\le\mathcal{M}\mathcal{L} \tag{10}$$

for every $\nu\in(0,\nu^*)$. Since x_ν is a critical point of I_ν, we obtain $I'_\nu(x_\nu)(y)=0$ for any $y\in\mathcal{B}$ and all $\nu\in(0,\nu^*)$. In particular, $I'_\nu(x_\nu)(x_\nu)=0$, that is,

$$\mathcal{J}_1'(x_\nu)(x_\nu)=\nu\int_0^{\sigma(T)} f(t,x_\nu^\sigma(t))x_\nu^\sigma(t)\Delta t \tag{11}$$

for every $\nu\in(0,\nu^*)$. Then, since

$$0\le\|x_\nu\|_0^2\le\mathcal{J}_1'(x_\nu)(x_\nu),$$

by using (11), it is concluded that

$$0\le\|x_\nu\|_0^2\le\mathcal{J}_1'(x_\nu)(x_\nu)\le\nu\int_0^{\sigma(T)} f(t,x_\nu^\sigma(t))x_\nu^\sigma(t)\Delta t$$

for any $\nu\in(0,\nu^*)$. Letting $\nu\to 0^+$, by (10), we have $\lim_{\nu\to 0^+}\|x_\nu\|_0=0$. One has

$$\lim_{\nu\to 0^+}\|x_\nu\|_\infty=0. \tag{12}$$

Finally, we demonstrate that the map

$$\nu\mapsto I_\nu(x_\nu)$$

strictly decreases in $(0,\nu^*)$. To do so, we note that

$$I_\nu(x)=\nu\left(\frac{\mathcal{J}_1(x)}{\nu}-\mathcal{J}_2(x)\right)\quad\text{for all}\quad x\in\mathcal{H}. \tag{13}$$

Fix $0<\nu_1<\nu_2<\nu^*$ and let x_{ν_1}, x_{ν_2} be the global minima of I_{ν_1}, I_{ν_2}, restricted to $\mathcal{J}_1(-\infty,s)$. Put

$$m_{\nu_i}=\frac{\mathcal{J}_1(x_{\nu_i})}{\nu_i}-\mathcal{J}_2(x_{\nu_i})=\inf_{y\in\mathcal{J}_1^{-1}(-\infty,s)}\left(\frac{\mathcal{J}_1(y)}{\nu_i}-\mathcal{J}_2(y)\right)$$

for $i=1,2$. Then, (9) and (13), since $\nu>0$, yield

$$m_{\nu_i}<0 \text{ for } i=1,2. \tag{14}$$

Moreover,

$$m_{\nu_2}\le m_{\nu_1} \tag{15}$$

due to $0<\nu_1<\nu_2$. Then by considering (13)–(15) and again since $0<\nu_1<\nu_2$, we get

$$I_{\nu_2}(x_{\nu_2})=\nu_2 m_{\nu_2}\le\nu_2 m_{\nu_1}<\nu_1 m_{\nu_1}=I_{\nu_1}(x_{\nu_1}),$$

so that the map $\nu\mapsto I_\nu(x_\nu)$ strictly decreases in $\nu\in(0,\nu^*)$. As $\nu<\nu^*$ is arbitrary, $\nu\mapsto I_\nu(x_\nu)$ strictly decreases in $(0,\nu^*)$.

Remark 5. We remark that Theorem 3 is a bifurcation result, i.e., (0,0) belongs to the closure of

$$\left\{(x_\nu,\nu)\in\mathcal{H}\times(0,\infty):\ x_\nu \text{ is a nontrivial solution of } (P_\nu^f)\right\}$$

in $\mathcal{H}\times\mathbb{R}$. Indeed, we know that

$$\|x_\nu\|_0\to 0\quad\text{as}\quad\nu\to 0.$$

Thus, there are two sequences $\{x_j\}$ in $\mathcal{H}$ and $\nu>0$ (here $x_j=x_\nu$) such that

$$\nu_i\to 0^+\quad\text{and}\quad\|x_j\|_0\to 0$$

as $j\to\infty$. Additionally, we point out that since the map

$$(0,\nu^*)\ni\nu\mapsto I_\nu(x_\nu)$$

is strictly decreasing, for every $\nu_1, \nu_2 \in (0, \nu^*)$ with $\nu_1 \neq \nu_2$, the solutions x_{ν_1} and x_{ν_2} are different.

Remark 6. If $f \geq 0$, then also the solution in Theorem 3 is nonnegative. To show this, assume x_0 is a nontrivial solution of (P_ν^f). Suppose

$$\mathcal{A} = \{t \in (0, T]_{\mathbb{T}} : \ x_0(t) < 0\}$$

has positive measure. Define $\bar{y}(t) = \min\{0, x_0(t)\}$ for $t \in [0, T]_{\mathbb{T}}$. We obtain $\bar{y} \in \mathcal{H}$ and

$$\int_0^{\sigma^2(T)} p(t)x_0^\Delta(t)\bar{y}^\Delta(t)\Delta t + \int_0^{\sigma(T)} q(t)x_0^\sigma(t)\bar{y}^\sigma(t)\Delta t + \beta_1 p(0)x_0(0)\bar{y}(0)$$
$$+ \beta_2 p(\sigma(T))x_0(\sigma^2(T))\bar{y}(\sigma^2(T)) - \nu\int_0^{\sigma(T)} f(t, x_0^\sigma(t))\bar{y}^\sigma(t)\Delta t = 0.$$

Thus, since f is assumed to be nonnegative, we get

$$\begin{aligned} 0 \leq \|x_0\|_{\mathcal{A}}^2 &\leq \int_0^{\sigma^2(T)} p(t)x_0^\Delta(t)x_0^\Delta(t)\Delta t + \int_0^{\sigma(T)} q(t)x_0^\sigma(t)x_0^\sigma(t)\Delta t \\ &+ \beta_1 p(0)x_0(0)x_0(0) + \beta_2 p(\sigma(T))x_0(\sigma^2(T))x_0(\sigma^2(T)) \\ &= \nu\int_0^{\sigma(T)} f(t, x_0^\sigma(t))x_0^\sigma(t)\Delta t \leq 0. \end{aligned}$$

Hence, $x_0 = 0$ in $\mathcal{A}$, and this is a contradiction.

The next theorem is concerned with a particular case of our results.

Theorem 4. *Assume $f : \mathbb{R} \to [0, \infty)$ is continuous and define $F(\xi) = \int_0^\xi f(s)\mathrm{d}s$ for $\xi \in \mathbb{R}$. If*

$$\lim_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} = \infty,$$

then, for all

$$\nu \in \Lambda = \left(0, \frac{1}{2\sigma(T)C^2} \sup_{\theta > 0} \frac{\theta^2}{F(\theta)}\right),$$

the problem

$$\begin{cases} -(px^\Delta)^\Delta(t) + q(t)x^\sigma(t) = \nu f(x^\sigma(t)), & t \in [0, T]_{\mathbb{T}}, \\ \alpha_1 x(0) - \alpha_2 x^\Delta(0) = 0, & \alpha_3 x(\sigma^2(T)) + \alpha_4 x^\Delta(\sigma(T)) = 0 \end{cases}$$

has a nontrivial solution $x_\nu \in \mathcal{H}$ satisfying

$$\lim_{\nu \to 0^+} \|x_\nu\|_0 = 0,$$

and

$$\nu \mapsto \frac{1}{2}\|x\|_0^2 - \nu\int_0^{\sigma(T)} F(x(t))\Delta t$$

is strictly decreasing in Λ and negative.

Finally, we offer an example illustrating Theorem 4.

Example 1. Let $\mathbb{T} = \left\{\frac{4}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ and $T = 1$. Consider

$$\begin{cases} -x^{\Delta\Delta}(t) = \nu f(x^\sigma(t)), & t \in [0, 1]_{\mathbb{T}}, \\ x(0) - 2x^\Delta(0) = 0, & x^\Delta(\frac{4}{3}) = 0, \end{cases} \tag{16}$$

where

$$f(\xi) = \frac{3}{64}\left(2\xi + 2\tan(\xi)\sec^2(\xi) + e^\xi\right) \quad \text{for all} \quad \xi \in \mathbb{R}.$$

Then (16) is in the form of (P_ν^f) with

$$\alpha_1 = 1, \quad \alpha_2 = 2, \quad \alpha_3 = 0, \quad \alpha_4 = 1, \quad p(t) \equiv 1, \quad q(t) \equiv 0.$$

We calculate

$$\beta_1 = \frac{1}{2}, \quad \beta_2 = 0, \quad \sigma(1) = \frac{4}{3}, \quad \sigma^2(1) = 2,$$

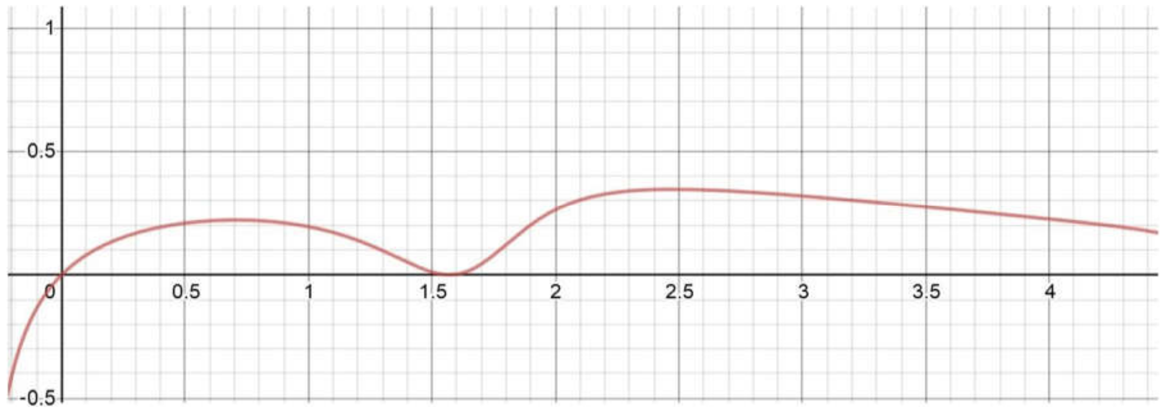

Fig. 1. Graph of $\frac{\theta^2}{\theta^2+\tan^2(\theta)+e^\theta-1}$.

$$\underline{p}=1,\quad \underline{q}=0,\quad M_1=M_2=2,\quad M_3=\infty,\quad C=2,$$

and

$$F(\xi)=\frac{3}{64}\left(\xi^2+\tan^2(\xi)+e^\xi-1\right)\quad \text{for all}\quad \xi\in\mathbb{R}.$$

Note that we clearly have (using L'Hôpital's rule)

$$\lim_{\xi\to 0^+}\frac{F(\xi)}{\xi^2}=\infty.$$

Hence, all assumptions in Theorem 4 are fulfilled. Note that

$$\sup_{\theta>0}\frac{\theta^2}{\theta^2+\tan^2(\theta)+e^\theta-1}\approx 0.347529\geq 0.3475$$

(see also Fig. 1) so that, by Theorem 4, (16), for all $\nu\in(0,\,0.695)$, has a nontrivial solution $x_\nu\in \mathrm{H}^1_\Delta([0,\sigma^2(1)]_\mathbb{T})$ satisfying

$$\lim_{\nu\to 0^+}\|x_\nu\|_0=0,$$

and the map

$$\nu\mapsto\frac{1}{2}\|x\|_0^2-\nu\int_0^{\frac{4}{3}}F(x(t))\Delta t$$

is strictly decreasing in (0,0.695) and negative.

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