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Granular fuzzy calculus on time scales and its applications to fuzzy dynamic equations

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ABSTRACT

This paper introduces the foundational theory of fuzzy calculus on time scales, utilizing granular arithmetic operations between fuzzy intervals. These operations are developed based on the concept of the horizontal membership function (HMF), which is applied in multidimensional fuzzy arithmetic (MFA). Furthermore, the paper explores the existence of a unique solution and the continuous dependence of the solution to fuzzy dynamic equations on initial data, employing the Banach fixed-point theorem under a new metric for fuzzy functions in time scales involving the generalized exponential function. Finally, to highlight the practical significance of these results and their potential applications, the paper presents mathematical models relevant to nuclear physics and biology.

1. Introduction

Time scales calculus and dynamic equations on time scales, initially proposed by S. Hilger, have garnered significant attention due to their applications in various fields in pure and applied mathematics [25,45]. This approach enables the investigation of dynamic systems within a unified framework, avoiding the need to study discrete and continuous components separately. Moreover, in practice, time scales go beyond merely unifying discrete and continuous analysis; they serve as an effective tool for studying dynamic systems on more complex and generalized domains that are neither continuous nor uniformly discrete [21]. The study of discrete systems, along with more general time scale systems, has often proven to be a more realistic approach in many cases. This enables a wide range of applications across diverse fields such as physics, engineering, and economics, where systems often exhibit both continuous and discrete behaviors. For example, quantum calculus [24], a form of time-scale calculus, is employed in quantum mechanics to analyze systems that evolve over both continuous and discrete domains. In engineering, networks of dynamic multi-agent systems consist of agents whose information exchange is neither purely continuous nor discrete, but instead varies as a function of time (see [15] and

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references therein). Similarly, in a traffic system, cars move smoothly along roads but are regulated by traffic lights that switch at discrete intervals [39]. Classical economic models developed on non-standard time scales show that time-scale analysis can explain phenomena in behavioral economics, particularly those involving intertemporal choices [16]. In population dynamics, the discrete version of the logistic growth model, known as the Beverton-Holt equation, has been regarded as a more realistic model since the seminal work of R. J. H. Beverton and S. J. Holt [5]. For recent advancements regarding the Beverton-Holt equation on time scales, readers can refer to the paper [8].

Several studies have explored both the qualitative and quantitative characteristics of solutions to dynamic equations on time scales. A study by C. C. Tisdell and Zaidi [41] examined the existence of a unique solution to first-order differential equations using the Banach and Schaefer fixed-point theorems. Additionally, successive approximations of the solution and the continuity of solutions to dynamic equations with their respective initial values were discussed. Some generalized results for first-order dynamic equations were recently introduced by M. Bohner et al. [10,40], and I. L. D. dos Santos [35]. For more in-depth insights into integral and functional dynamic equations and some related problems on time scales, readers can refer to the books [6,32].

Example 1.1. Dynamic equations on time scales can be particularly useful in real-world scenarios where processes do not occur continuously or solely at discrete moments. Recently, the applications of dynamic equations on time scales in surveying the practical models have been demonstrated in the work of R. Agarwal et al. [1] (see also references therein) for the models of population growth of plants and electric circuit. Similar to Example 4.3 of [1], we will consider a population of butterflies $X(z)$ at time z in a temperate region. The butterfly population grows exponentially according to the differential equation

$$X'(z) = pX(z), \quad X(0) = X_0, \tag{1.1}$$

during the warm months (e.g., from March to August), where p is the growth rate and X_0 represents the initial butterfly population size. As the temperature drops in September, the adult butterflies die off, leaving behind dormant pupae. These pupae hatch at the beginning of the next warm season, with the population size $X(z)$ being a fixed multiple (e.g., doubling) of the population at the end of the previous warm season. We can model this process using a time scale

$$\mathbb{T} = \bigcup_{m=0}^{\infty} [2m, 2m+1], \tag{1.2}$$

where $z = 0$ corresponds to March 1 of the current year (start of the growing season), $z = 1$ corresponds to September 1 of the current year (end of the growing season), $z = 2$ corresponds to March 1 of the next year (start of the next growing season), and so on. In (1.2), $[2m, 2m+1)$ represents the growing season (March to August) of the year m, and $(2m+1)$ represents the transition point (September 1) when the population resets and pupae remain dormant until the next growing season. On the time scale $\mathbb{T}$, the model (1.1) is expressed as the dynamic equation

$$X^{\Delta}(z) = pX(z), \quad X(0) = X_0, \tag{1.3}$$

which governs the growth of the population during the interval $[2m, 2m+1)$, where $X^{\Delta}(z)$ means the delta derivative of X at z. Furthermore, at the end of the growing season, specifically at $z = 2m+1$ (September 1), all adult butterflies die, and the population resets to a new value based on the dormant pupae. The population size at the beginning of the next growing season (March 1 of the next year, $z = 2m+2$) is a multiple of the population at the end of the previous growing season. For this example, let us assume the population doubles: $X(2m+2) = 2X(2m+1)$. This relationship is captured by the difference equation: $\Delta X(2m+1) = X(2m+1)$, which implies that the population doubles at the transition points $z = 2m+1$. Consequently, given the initial condition and the dynamic equation (1.3), the solution for $X(z)$ on the time scale $\mathbb{T}$ is given by the time scale exponential function: $X(z) = X_0 e_p(\cdot, 0)$, where $e_p(\cdot, 0)$ represents the time scale exponential function that describes the population's growth behavior during the growing season.

Observe that this model can be applied to various insect populations that exhibit similar seasonal growth patterns. It is particularly useful for studying populations in regions with distinct seasons, where environmental factors like temperature play a significant role in population dynamics. By using time scales calculus, this model unifies the continuous growth during the active season with the discrete population changes between seasons, providing a more realistic representation of the insect population dynamics.

In practical scenarios, dynamic equations designed to model natural phenomena often face uncertainties and vagueness. These uncertainties typically arise from imprecise measurements or the inherent indeterminacy of events. To address these challenges, the classical theory of dynamic equations has been extended to include frameworks such as interval dynamic equations and fuzzy dynamic equations. Applications of uncertain (interval or fuzzy) dynamic equations on time scales span a variety of fields, including control theory, optimization, hybrid system modeling (which combines continuous and discrete dynamics), and decision-making in uncertain environments. One of the most challenging tasks in the theory of uncertain differential equations (UDEs), including interval and fuzzy types, is the development of derivatives and integrals for fuzzy or interval functions, with a primary focus on defining operations involving fuzzy numbers or intervals. Consequently, several approaches have been developed to explore uncertain dynamic equations, which can be broadly categorized into four key directions: (1) The generalized Hukuhara difference (gH-difference) approach, studied by L. Stefanini et al. [38]. (2) The linearly correlated difference approach, explored by E. Esmi et al. [14]. (3) The method utilizing the difference between fuzzy intervals through constraint interval arithmetic, presented in the works of W. A. Lodwick [22]. (4) The granular difference (gr-difference) approach, defined using the horizontal membership function (HMF) in multidimensional fuzzy arithmetic (MFA), studied by A. Piegat et al. [33] and M. Mazandarani et al. [27]. These approaches have significantly contributed to the ongoing exploration of UDEs and the theory of uncertain dynamic equations. By offering diverse perspectives and methodologies, they address the uncertainties in dynamic equations and enhance both the understanding and practical application of these concepts in real-world scenarios.

Concerning the first approach, based on the gH-difference, numerous research efforts have been dedicated to the theoretical development of uncertain dynamic equations on time scales. Specifically, S. Hong proposed the delta Hukuhara derivative and delta integral for multivalued functions on arbitrary time scales in [19], utilizing the Hukuhara difference and investigating the initial value problem of set integro-dynamic equations. In [23], V. Lupulescu introduced novel concepts for the delta derivative and the Riemann integral of interval functions on arbitrary time scales, involving the gH-difference and the forward jump operator. An interval dynamic equation on a time scale was investigated as an application of these concepts. Furthermore, Ch. Vasavi et al. [43] proposed generalized differentiability and integrability for fuzzy functions on time scales and explored some of their fundamental properties. In a related vein, the authors in [42] presented a comprehensive survey on partial delta derivatives for binary fuzzy functions using the gH-difference, applying this concept to analyze the fuzzy transport equation on time scales. Ongoing research has produced numerous contributions to time scales calculus involving interval-valued functions (IVFs) and fuzzy functions, as demonstrated in works such as [20,32] and the references therein. These studies deepen the understanding of uncertain dynamic equations and the methods used for their analysis. However, this approach has several limitations, including the catastrophe of physics laws violation (CPLV), unnatural behavior in modeling (UBM), the presence of multiple solutions, and challenges in accurately representing real-world problems. These issues have been thoroughly examined by M. Mazandarani et al. [27,28], who identified at least six specific limitations associated with the use of the gH-difference approach in investigating uncertain dynamic equations.

The second approach has proven to be more effective and reliable than the first. Significant recent advancements and applications in this area are evident in the study of UDEs within a fuzzy framework. Recently, L.C. de Barros et al. introduced delta derivatives and integrals involving correlated fuzzy processes on time scales, as detailed in [36]. Using this concept, the authors analyzed fuzzy Volterra-type integral equations and established a connection between fuzzy initial value problems and fuzzy Volterra-type integral equations with correlated fuzzy processes on hybrid domains. However, we note that this approach can be complex and challenging when applied to uncertain dynamic equations.

The third approach utilizes constraint fuzzy arithmetic (CFA), an extension of interval arithmetic that aims to restore the algebraic properties of real arithmetic that are often lacking in traditional interval arithmetic. By addressing issues such as the absence of an additive inverse, multiplicative inverse, and distributive law, CFA offers a more robust and flexible method. However, this approach does not always guarantee that the results of computations will yield fuzzy numbers, as discussed in Section 5 of [34]. To address this limitation, Y. Chalco-Cano et al. [12] proposed defining fuzzy arithmetic operations using the generalized single-level CFA approach. For recent applications of the CFA method, we recommend the work by M. S. Cecconello et al. [11], in which CFA is applied to study the spread of the SARS-CoV-2 virus.

The remaining approach in this field utilizes the HMF concept in MFA to develop fuzzy arithmetic operations. This method has proven effective in addressing the limitations of previous approaches. For more details, readers can refer to [28]. Recently, the authors in [27] introduced a new concept of fuzzy derivative for continuous-time domains, known as the granular derivative (gr-derivative). This approach has shown greater reliability and advantages compared to earlier methods, particularly in addressing the challenges associated with the gH-derivative concept. Consequently, this method has opened new avenues for research in fuzzy analysis and fuzzy differential equations. Significant contributions have been made in this area, including studies on the existence and uniqueness of fuzzy delay differential equations [13], fuzzy optimal control problems [29], stability and controllability in fuzzy singular dynamical systems [31], and fuzzy fractional differential equations [3,4,17,30]. However, the development of uncertain calculus on time scales and uncertain dynamic equations using granular arithmetic operations has received limited attention. This area remains largely unexplored, with no dedicated studies conducted to date. We believe that developing uncertain calculus on time scales using this approach presents significant potential for advancing uncertainty theory in dynamic equations. Therefore, the goal of this paper is to introduce new concepts of differentiability and integrability for functions on general time scales within a fuzzy environment, based on granular arithmetic operations. Our aim is to provide a robust tool for analyzing fuzzy differential equations on time scales. The key contributions of this study are summarized as follows:

i) We establish the concept of the limit of fuzzy functions on time scales by employing a granular metric on the fuzzy number set. This limit enables us to provide an equivalent definition of the derivative of fuzzy functions using limit-based language.
ii) We introduce the granular delta derivative and granular delta integral for fuzzy functions on time scales, building upon the concepts of granular difference and limit. Several key characteristics of these concepts are rigorously demonstrated.
iii) We achieve the existence of a unique solution to fuzzy dynamic equations with initial conditions on an arbitrary time scale. Moreover, we investigate the continuity of solutions to fuzzy dynamic equations concerning variations in initial values.
iv) To demonstrate the practical significance of our findings, we present models in nuclear physics and biology that pertain to hybrid domains.

The remaining sections of this paper are structured as follows: In Section 2, we revisit crucial definitions and preliminary results from the calculus of time scales and fuzzy arithmetic, which are utilized in the subsequent sections. Then, in Section 3, we introduce fuzzy calculus on time scales, including the concepts of limit, granular delta differentiability, and the granular integral of fuzzy functions. Section 4 is dedicated to investigating the qualitative and quantitative characteristics of fuzzy differential equations on time scales. The final section provides applications in the fields of nuclear physics and biology.

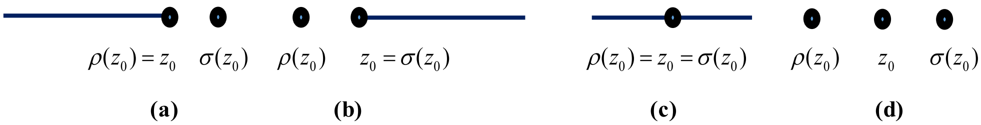

Fig. 1. (a) z_0 is right-scattered and left-dense. (b) z_0 is right-dense and left-scattered. (c) z_0 is dense. (d) z_0 is isolated.

2. Preliminaries

In what follows, we adopt the notations $\mathbb{Z}$ and $\mathbb{R}$ to indicate the families of all integer and real numbers, respectively.

2.1. Time scales essentials

For the reader's convenience, we provide a brief overview of time scales calculus. Additional details can be found in [9]. Consider a time scale denoted as $\mathbb{T}$, defined as a nonempty, closed subset of the real numbers $\mathbb{R}$. Examples of such time scales include the discrete set $h\mathbb{Z}$ with a fixed step size $h > 0$, the set of all natural numbers $\mathbb{N}$, the entire real number line $\mathbb{R}$, as well as any discrete subset or a combination of discrete points and closed intervals.

For a given time scale $\mathbb{T}$ and a point $z_0 \in \mathbb{T}$, the *forward* and *backward jump operators* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are respectively defined as

$$\sigma(z_0) := \inf\{z \in \mathbb{T} \mid z > z_0\} \quad \text{and} \quad \rho(z_0) := \sup\{z \in \mathbb{T} \mid z < z_0\}.$$

If $\sup(\mathbb{T}) = C < \infty$, then $\sigma(C) = C$. A point $z_0 \in \mathbb{T}$ is called *right-scattered, left-scattered, right-dense*, and *left-dense* if $\sigma(z_0) > z_0$, $\rho(z_0) < z_0$, $\sigma(z_0) = z_0$, and $\rho(z_0) = z_0$, respectively. A graininess function is a real function $\mu : \mathbb{T} \to \mathbb{R}^+$ given by $\mu(z_0) = \sigma(z_0) - z_0$, where $z_0 \in \mathbb{T}$ and μ is considered as the distance between two consecutive points. Observe that the function $\mu(z) = 0$ if $z \in \mathbb{T} = \mathbb{R}$ and $\mu(z) = 1$ for $z \in \mathbb{T} = \mathbb{Z}$. A time scale $\mathbb{T}$ is said to be *isolated* if all of elements are both left and right-scattered. Fig. 1 represents the forward and backward jump operators in different special cases. In this work, for some $a, b \in \mathbb{T}$, we denote the time scale interval $[a, b]_{\mathbb{T}}$ by the intersection of the real interval $[a, b]$ with $\mathbb{T}$. Other time scale intervals such as $[a, b)_{\mathbb{T}}$, $(a, b]_{\mathbb{T}}$, and $(a, b)_{\mathbb{T}}$ can be defined analogously. For a given time scale $\mathbb{T}$, we define the *kappa set* $\mathbb{T}^\kappa$ of $\mathbb{T}$ as follows

$$\mathbb{T}^\kappa = \mathbb{T} \quad \text{if} \quad \sup \mathbb{T} = \infty \qquad \text{and} \qquad \mathbb{T}^\kappa = \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}] \quad \text{if} \quad \sup \mathbb{T} < \infty.$$

For a given point $z_0 \in \mathbb{T}$ and any $\delta > 0$, the neighborhood $U_{\mathbb{T}}(z_0, \delta)$ of z_0 on time scale $\mathbb{T}$ is defined as $U_{\mathbb{T}}(z_0, \delta) = (z_0 - \delta, z_0 + \delta)_{\mathbb{T}}$.

Definition 2.1. Let $\phi : \mathbb{T} \to \mathbb{R}$ be a real function and let $z_0 \in \mathbb{T}^\kappa$ be given. Function ϕ is said to be *delta differentiable* (Δ*-differentiable*) at z_0 provided there exists a real number $\phi^\Delta(z_0)$ with the property that for any $\varepsilon > 0$, there is $\delta > 0$ satisfying that

$$|\phi(\sigma(z_0)) - \phi(z) - \phi^\Delta(z_0)(\sigma(z_0) - z)| \le \varepsilon |\sigma(z_0) - z| \qquad \text{for all } z \in U_{\mathbb{T}}(z_0, \delta). \tag{2.1}$$

The real number $\phi^\Delta(z_0)$ is called the *delta derivative* (Δ*-derivative*) of ϕ at z_0. If ϕ is Δ-differentiable at any $z \in \mathbb{T}^\kappa$, then we say ϕ is Δ-differentiable on $\mathbb{T}^\kappa$.

Some important characteristics of delta differentiability are listed in the below theorem.

Theorem 2.1. *(M. Bohner et al. [9]) Consider real functions $\phi, \psi : \mathbb{T} \to \mathbb{R}$ and $z_0 \in \mathbb{T}^\kappa$. If ϕ, ψ are Δ-differentiable at z_0, then*

i) *so is $\phi + \psi$ and $(\phi + \psi)^\Delta(z_0) = \phi^\Delta(z_0) + \psi^\Delta(z_0)$,*
ii) *so is $\beta\phi$ and $(\beta\phi)^\Delta(z_0) = \beta\phi^\Delta(z_0)$ for any $\beta \in \mathbb{R}$,*
iii) *so is $\phi\psi$ and $(\phi\psi)^\Delta(z_0) = \phi^\Delta(z_0)\psi(z_0) + \phi(\sigma(z_0))\psi^\Delta(z_0)$,*
iv) *if $\psi(z) \neq 0$ on $[z_0, \sigma(z_0)]_{\mathbb{T}}$, then ϕ/ψ is Δ-differentiable at z_0 and*

$$(\phi/\psi)^\Delta(z_0) = [\phi^\Delta(z_0)\psi(z_0) - \phi(z_0)\psi^\Delta(z_0)]/[\psi(z_0)\psi(\sigma(z_0))].$$

Let $\phi : \mathbb{T} \to \mathbb{R}$ be a real function. If ϕ exhibits continuity at right-dense points in $\mathbb{T}$ and finite left-sided limits exist at left-dense points of $\mathbb{T}$, then the function ϕ is termed as *right-dense* continuous. The sets of all right-dense continuous functions and continuous functions are denoted by $C_{\rm rd}(\mathbb{T}, \mathbb{R})$ and $C(\mathbb{T}, \mathbb{R})$, respectively. A real-valued function $\Psi : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of ϕ if $\Psi^\Delta(z) = \phi(z)$ for all $z \in \mathbb{T}^\kappa$. We notice that function ϕ has an antiderivative if $\phi \in C_{\rm rd}(\mathbb{T}, \mathbb{R})$.
Based on the concept of the antiderivative of ϕ, the delta integral of ϕ is given in the below definition.

Definition 2.2. The *delta integral* (delta Cauchy integral) of a function $\phi : \mathbb{T} \to \mathbb{R}$ on $[a, b]_{\mathbb{T}}$, where $a, b \in \mathbb{T}$, is defined by

$$\int_a^b \phi(z)\Delta z = \Psi(b) - \Psi(a), \tag{2.2}$$

where $\Psi : \mathbb{T} \to \mathbb{R}$ is an antiderivative of ϕ.

Let $\phi : \mathbb{T} \to \mathbb{R}$ be a real function. We say that the function ϕ is *regressive* (or *positively regressive*) if $1 + \mu(z)\phi(z) \neq 0$ (or $1 + \mu(z)\phi(z) > 0$) for all $z \in \mathbb{T}^\kappa$. We denote $\mathcal{R}(\mathbb{T}, \mathbb{R})$ and $\mathcal{R}^+(\mathbb{T}, \mathbb{R})$ as the sets of all rd-continuous regressive and positively regressive functions, respectively. For $z_0 \in \mathbb{T}$ and $\phi \in \mathcal{R}(\mathbb{T}, \mathbb{R})$, the *generalized exponential function* $e_\phi(\cdot, z_0)$ on $\mathbb{T}$ is a solution to the initial value problem

$$\begin{cases} x^\Delta(z) = \phi(z)x(z), \\ x(z_0) = 1. \end{cases} \tag{2.3}$$

Explicitly, the exponential functions can be expressed by the formula

$$e_\phi(z, z_0) = \begin{cases} \exp\left(\int_{z_0}^{z} \frac{1}{\mu(s)} \mathrm{Log}(1 + \mu(s)\phi(s))\Delta s\right) & \text{if } \mu > 0, \\ \exp\left(\int_{z_0}^{z} \phi(s)\mathrm{d}s\right) & \text{if } \mu = 0. \end{cases} \tag{2.4}$$

2.2. Fundamentals of fuzzy arithmetic

The concepts of the horizontal membership function (HMF) and relative distance measure (RDM) in fuzzy interval analysis and multidimensional fuzzy arithmetic (MFA) were initially presented in the pioneering work of A. Piegat et al. [33]. A more comprehensive exploration of this concept can be found in the research by M. Mazandarani et al. [27]. This work primarily focuses on the development of calculus tools, such as the fuzzy granular derivative and fuzzy granular integral, while also investigating fuzzy differential equations within the context of continuous domains. Next, we recall a brief introduction to fuzzy numbers and their level sets.

Let $\omega : \mathbb{R} \to [0, 1]$. If ω fulfills conditions including upper semi-continuity, normality, fuzzy convexity, and closure of compact support, then ω is referred to as a fuzzy number on the real line. We represent the family of all fuzzy numbers as $\mathscr{F}_\mathbb{R}$. The r-level set of $\omega \in \mathscr{F}_\mathbb{R}$, denoted by $[\omega]^r$ for all $r \in [0, 1]$, is given as

$$[\omega]^r = \{x \in \mathbb{R} \mid \omega(x) \geq r\},$$

where $r \in (0, 1]$, and $[\omega]^0 := \overline{\{x \in \mathbb{R} \mid \omega(x) > 0\}}$, where $r = 0$. It is well known that $[\omega]^r$ is expressed by a compact interval, specifically, $[\omega]^r = [\underline{\omega}^r, \overline{\omega}^r]$, which is also referred to as a fuzzy interval. Additionally, the real functions $\underline{\omega}^r$, $\overline{\omega}^r : [0, 1] \to \mathbb{R}$ are known as the left and right borders of ω, respectively.

Below, we provide a summary overview of the HMF concept and the essential foundations associated with fuzzy arithmetic operations.

Definition 2.3. Let $\omega : [c, d] \subseteq \mathbb{R} \to [0, 1]$ be a given fuzzy number with the r-level set $[\omega]^r = [\underline{\omega}^r, \overline{\omega}^r], r \in [0, 1]$. The HMF of ω is defined as a real mapping $\omega^{\mathrm{gr}} : [0, 1] \times [0, 1] \to [c, d]$ given by

$$(r, \alpha_\omega) \mapsto \omega^{\mathrm{gr}}(r, \alpha_\omega) := \underline{\omega}^r + d([\omega]^r)\alpha_\omega, \tag{2.5}$$

where $d([\omega]^r) := (\overline{\omega}^r - \underline{\omega}^r)$ and $r \in [0, 1]$ is the membership grade of x in the fuzzy set ω and where $\alpha_\omega \in [0, 1]$ is called the RDM variable, which varies and is constrained between 0 and 1.

Remark 2.1. According to Definition 2.3, while the term "membership function" is part of the phrase "horizontal membership function", the real function of two variables ω^{gr} is not a classical membership function, and its values do not represent the degrees or grades of elements x in the fuzzy set ω. It is noticed that the HMF of ω is understood deeper through the following information:

- It is a real function of two variables that represents the classical (vertical) membership function. The HMF ω^{gr} is illustrated in a three-dimensional space (r, α_ω coordinate).
- Its values $\omega^{\mathrm{gr}}(r, \alpha_\omega) = x \in [c, d]$ gives an element that is in the fuzzy set ω.
- The notation α_ω is said to be the RDM variable, which is considered a horizontal index allowing us to access or scan the interior region of the (classical) membership function.

For convenience, we denote the representation of the fuzzy number ω in terms of its HMF as $\mathcal{H}(\omega) = \omega^{\mathrm{gr}}(r, \alpha_\omega)$. Additionally, the r-level sets of ω derived from the HMF are computed using the formula

$$\mathcal{H}^{-1}(\omega^{\mathrm{gr}}(r, \alpha_\omega)) = [\omega]^r = \left[\inf_{\beta \geq r} \min_{\alpha_\omega \in [0,1]} \omega^{\mathrm{gr}}(\beta, \alpha_\omega), \sup_{\beta \geq r} \max_{\alpha_\omega \in [0,1]} \omega^{\mathrm{gr}}(\beta, \alpha_\omega)\right]. \tag{2.6}$$

Definition 2.4. Consider $\omega_1, \omega_2 \in \mathscr{F}_\mathbb{R}$ with $\mathcal{H}(\omega_1)$ and $\mathcal{H}(\omega_2)$ representing the HMFs of ω_1 and ω_2, respectively. We denote one of the four fundamental operations on $\mathscr{F}_\mathbb{R}$ by $\circledast_{\mathrm{gr}}$, which include additive, subtractive, multiplicative, and divisive operations. Then, $\omega_1 \circledast_{\mathrm{gr}} \omega_2$ is a fuzzy number ω such that

$$\mathcal{H}(\omega) = \mathcal{H}(\omega_1) * \mathcal{H}(\omega_2), \tag{2.7}$$

where "$*$" stands for the four corresponding operations of $\circledast_{\rm gr}$ on $\mathbb{R}$. We assume that $0 \notin \mathcal{H}(\omega_2)$ if $\circledast_{\rm gr}$ is the divisive operation. Note that arithmetic operations between fuzzy numbers, conducted using their HMFs, will be referred to as granular operations.

The subtraction of ω_1 and ω_2 in Definition 2.4 is called the *granular difference*, denoted by $\omega_1 \ominus_{\rm gr} \omega_2$. To simplify notation, we use $\omega_1\omega_2$ for granular multiplication instead of $\omega_1 \odot_{\rm gr} \omega_2$ when no further distinctions are needed.

Remark 2.2. It is observed that granular operations can restore the algebraic properties of real arithmetic that are lacking in traditional fuzzy arithmetic. In particular, the following properties hold (see [27]): (i) $\omega_1 \ominus_{\rm gr} \omega_2 = \hat{0}$, where $\hat{0} \in \mathscr{F}_{\mathbb{R}}$ means the zero fuzzy number. (ii) $\omega_1 \ominus_{\rm gr} \omega_2 = \ominus_{\rm gr}(\omega_2 \ominus_{\rm gr} \omega_1)$. (iii) $(\omega_1 \oplus_{\rm gr} \omega_2) \odot_{\rm gr} \omega_3 = \omega_1 \odot_{\rm gr} \omega_3 \oplus_{\rm gr} \omega_2 \odot_{\rm gr} \omega_3$, where $\omega_1, \omega_2, \omega_3 \in \mathscr{F}_{\mathbb{R}}$.

Example 2.1. Let us consider triangular fuzzy numbers $\omega_1 = (1, 2, 4)$ and $\omega_2 = (0.5, 2.5, 3)$. For $0 \le r \le 1$, the r-level sets of ω_1 and ω_2 are given by

$$[\omega_1]^r = [1 + r, 4 - 2r] \quad \text{and} \quad [\omega_2]^r = [0.5 + 2r, 3 - 0.5r].$$

From Definition 2.3, the HMFs of ω_1 and ω_2 are expressed by $\mathcal{H}(\omega_1) = \omega_1^{\rm gr}(r, \alpha_{\omega_1}) = 1 + r + (3 - 3r)\alpha_{\omega_1}$ and $\mathcal{H}(\omega_2) = 0.5 + 2r + (2.5 - 2.5r)\alpha_{\omega_2}$ for all $\alpha_{\omega_1}, \alpha_{\omega_2} \in [0, 1]$ (see Fig. 2). From Definition 2.4, the difference $\omega_1 \ominus_{\rm gr} \omega_2$ and the multiplication $\omega_1 \odot_{\rm gr} \omega_2$ are fuzzy numbers with

$$\mathcal{H}(\omega_1 \ominus_{\rm gr} \omega_2) = 0.5 - r + (3 - 3r)\alpha_{\omega_1} - (2.5 - 2.5r)\alpha_{\omega_2} \tag{2.8}$$

and

$$\mathcal{H}(\omega_1 \odot_{\rm gr} \omega_2) = [1 + r + (3 - 3r)\alpha_{\omega_1}][0.5 + 2r + (2.5 - 2.4r)\alpha_{\omega_2}], \tag{2.9}$$

where $\alpha_{\omega_1}, \alpha_{\omega_2} \in [0, 1]$. By using the formula (2.6), we receive the r-level sets of $\omega_1 \ominus_{\rm gr} \omega_2$ and $\omega_1 \odot_{\rm gr} \omega_2$, respectively, as

$$[\omega_1 \ominus_{\rm gr} \omega_2]^r = [-2 + 1.5r, 3.5 - 4r] \tag{2.10}$$

and

$$[\omega_1 \odot_{\rm gr} \omega_2]^r = [0.5 + 2.5r + 2r^2, 12 - 8r + r^2] \tag{2.11}$$

for all $r \in [0, 1]$. Moreover, the membership functions of $\omega_1 \ominus_{\rm gr} \omega_2$ and $\omega_1 \odot_{\rm gr} \omega_2$ are

$$(\omega_1 \ominus_{\rm gr} \omega_2)(x) = \begin{cases} \frac{2}{3}x + \frac{4}{3} & \text{if } x \in [-2, -0.5], \\ -\frac{1}{4}x + \frac{7}{8} & \text{if } x \in [-0.5, 3.5], \\ 0 & \text{otherwise} \end{cases} \tag{2.12}$$

and

$$(\omega_1 \odot_{\rm gr} \omega_2)(x) = \begin{cases} \frac{1}{8}(\sqrt{32x + 9} - 5) & \text{if } x \in [0.5, 5], \\ 4 - \sqrt{x + 4} & \text{if } x \in [5, 12], \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

Figs. 3 and 4 illustrate the membership functions of ω_1, ω_2 and their operations.

Remark 2.3. From Definition 2.3, we can see that the mapping $\mathcal{H}$ is linear (see [44]). As a result, for any ω_1, ω_2, ω_3, and $\omega_4 \in \mathscr{F}_{\mathbb{R}}$, the following properties are fulfilled:

i) $\mathcal{H}(a\omega_1 \oplus_{\rm gr} b\omega_2) = a\mathcal{H}(\omega_1) + b\mathcal{H}(\omega_2)$ for all $a, b \in \mathbb{R}$,
ii) $\mathcal{H}[(\omega_1 \oplus_{\rm gr} \omega_2) \ominus_{\rm gr} (\omega_3 \oplus_{\rm gr} \omega_1)] = \mathcal{H}(\omega_1 \ominus_{\rm gr} \omega_3) + \mathcal{H}(\omega_2 \ominus_{\rm gr} \omega_4)$.

The relations between two fuzzy numbers ω_1 and ω_2 are defined as follows.

Definition 2.5. Let $\omega_1, \omega_2 \in \mathscr{F}_{\mathbb{R}}$, and let $\mathcal{H}(\omega_1)$ and $\mathcal{H}(\omega_2)$ be their HMFs, respectively. If $\mathcal{H}(\omega_1) = \mathcal{H}(\omega_2)$ for all $r \in [0, 1]$ and $\alpha_{\omega_1} = \alpha_{\omega_2} \in [0, 1]$, we say that ω_1 is *equal* to ω_2. We say that fuzzy numbers ω_1 and ω_2 are in the relation $\omega_1 \preceq_{\rm gr} \omega_2$ if $\mathcal{H}(\omega_1) \le \mathcal{H}(\omega_2)$ for all $r \in [0, 1]$ and $\alpha_{\omega_1} = \alpha_{\omega_2} \in [0, 1]$.

The distance between fuzzy numbers based on their HMFs is called a granular metric. This distance is defined as

$$\mathfrak{D}_{\rm gr} : \mathscr{F}_{\mathbb{R}} \times \mathscr{F}_{\mathbb{R}} \quad \to \quad \mathbb{R}^+ \cup \{0\}$$

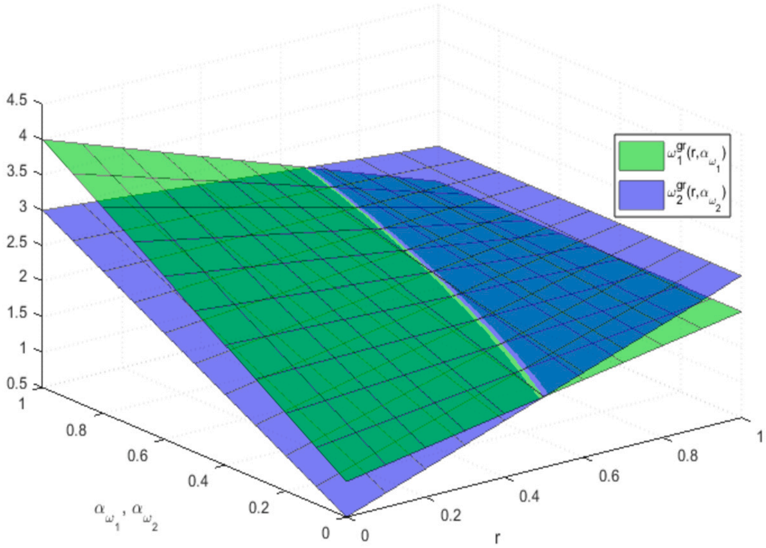

Fig. 2. The HMFs of the fuzzy numbers ω_1 (green) and ω_2 (blue).

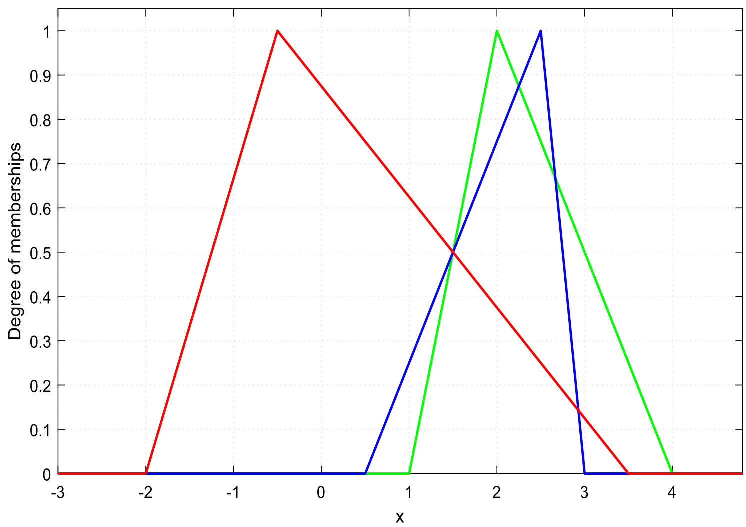

Fig. 3. The polylines represent the graphs of the membership functions of ω_1 (green), ω_2 (blue), and $\omega_1 \ominus_{gr} \omega_2$ (red).

$$(\omega_1, \omega_2) \quad \mapsto \quad \mathfrak{D}_{gr}(\omega_1, \omega_2) := \sup_{r\in[0,1]} \max_{\alpha_{\omega_1}, \alpha_{\omega_2} \in [0,1]} \left| \mathcal{H}(\omega_1) - \mathcal{H}(\omega_2) \right| \tag{2.14}$$

for any $\omega_1, \omega_2 \in \mathscr{F}_{\mathbb{R}}$. It is well known that the metric space $(\mathscr{F}_{\mathbb{R}}, \mathfrak{D}_{gr})$ is complete. For any $\omega_1, \omega_2, \omega_3, \omega_4 \in \mathscr{F}_{\mathbb{R}}$, the metric $\mathfrak{D}_{gr}$ has the following properties (see [27]):

i) $\mathfrak{D}_{gr}(\omega_1, \omega_2) = \mathfrak{D}_{gr}(\omega_2, \omega_1)$,
ii) $\mathfrak{D}_{gr}(c\omega_1, c\omega_2) = |c|\mathfrak{D}_{gr}(\omega_1, \omega_2)$, where c is a constant,
iii) $\mathfrak{D}_{gr}(\omega_1 \oplus_{gr} \omega_2, \omega_3 \oplus_{gr} \omega_4) \leq \mathfrak{D}_{gr}(\omega_1, \omega_3) + \mathfrak{D}_{gr}(\omega_2, \omega_4)$,
iv) $\mathfrak{D}_{gr}(\omega_1 \ominus_{gr} \omega_2, \omega_3 \ominus_{gr} \omega_4) \leq \mathfrak{D}_{gr}(\omega_1, \omega_3) + \mathfrak{D}_{gr}(\omega_2, \omega_4)$.

Remark 2.4. Recently, arithmetic operations between fuzzy numbers, explored through the generalized single-level constrained fuzzy arithmetic (CFA) and the HMF used in MFA, have been thoroughly analyzed and discussed by N. V. Hoa et al. [18]. As mentioned in [18, Subsection 2.2.3], although HMF might seem similar to the increasing constraint function in CFA and the generalized single-level CFA (see [12, Definition 9-(iii)]), their arithmetic operations are defined differently. Specifically, the HMF approach performs arithmetic operations with multiple α (RDM variable) parameters for all variables, while the generalized single-level CFA consistently applies the same constraint parameter across each fuzzy interval involved. As a result, the HMF-based method significantly influences

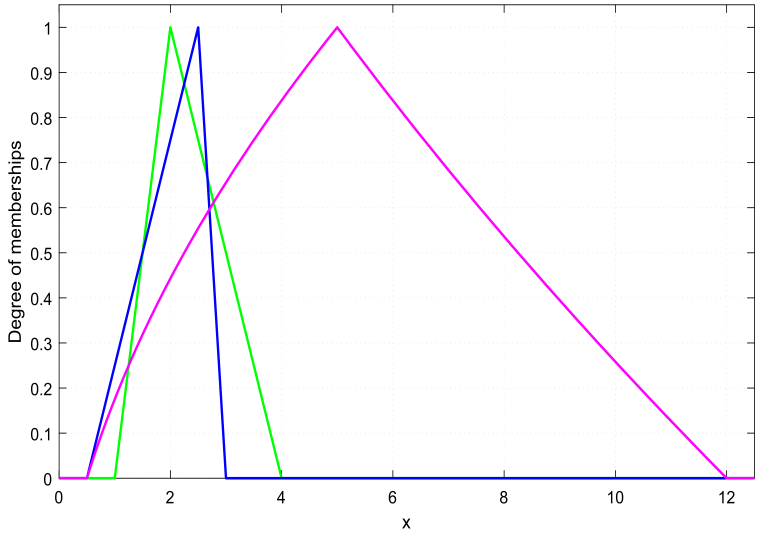

Fig. 4. The curves represent the graphs of the membership functions of ω_1 (green), ω_2 (blue), and $\omega_1 \odot_{gr} \omega_2$ (pink).

the calculation outcomes compared to other approaches. For additional clarification on this matter, readers can consult the work by N. V. Hoa et al. [18].

3. Time scales calculus for fuzzy functions

In this section, we introduce the extensions of delta differentiability and integrability on time scales through the HMFs of fuzzy functions. First, we establish the fundamental concepts of fuzzy calculus on time scales, including fuzzy functions and their limits.

Definition 3.1. A mapping Φ defined on $[a,b]_{\mathbb{T}}$ with values in $\mathscr{F}_{\mathbb{R}}$ is *a fuzzy function*. If Φ contains n distinct fuzzy numbers $\omega_1, \dots, \omega_n$, then the HMF Φ^{gr} of Φ at $z \in [a,b]_{\mathbb{T}}$, denoted by $\mathcal{H}(\Phi(z)) := \Phi^{gr}(z, r, \alpha_\Phi)$, is defined as

$$\begin{aligned}\Phi^{gr} : [a,b]_{\mathbb{T}} \times [0,1] \times [0,1]^n &\to \mathbb{R} \\ (z, r, \alpha_\Phi) &\mapsto \Phi^{gr}(z, r, \alpha_\Phi),\end{aligned} \tag{3.1}$$

where $r \in [0,1]$, and $\alpha_\Phi := (\alpha_{\omega_1}, \dots, \alpha_{\omega_n})$, with $\alpha_{\omega_1}, \dots, \alpha_{\omega_n} \in [0,1]$, are referred to as the corresponding relative distance measure variables.

Example 3.1. For a given $h > 0$, let us consider the time scale $\mathbb{T} = h\mathbb{Z}$, and a fuzzy function $\Phi : [0,2]_{h\mathbb{Z}} \to \mathscr{F}_{\mathbb{R}}$ defined by:

$$\Phi(z) = \tilde{1} e_{-p}(z, 0) \oplus_{gr} \tilde{2} z^2, \tag{3.2}$$

where $\tilde{1} = (0.75, 1, 1.25)$, $\tilde{2} = (1.5, 2, 2.25) \in \mathscr{F}_{\mathbb{R}}$, and $e_{-p}(z,0)$ with $-p \in \mathcal{R}(\mathbb{T}, \mathbb{R})$ is a generalized exponential function on arbitrary time scales. From Definition 2.3, we derive the HMFs of $\tilde{1}$ and $\tilde{2}$ as

$$\mathcal{H}(\tilde{1}) = 0.75 + 0.25r + (0.5 - 0.5r)\alpha_1 \quad \text{and} \quad \mathcal{H}(\tilde{2}) = 1.5 + 0.5r + (0.75 - 0.75r)\alpha_2 \tag{3.3}$$

with $\alpha_1, \alpha_2 \in [0,1]$. Then, the HMF of $\Phi(z)$ is as follows:

$$\Phi^{gr}(z, r, \alpha_\Phi) = [0.75 + 0.25r + (0.5 - 0.5r)\alpha_1] e_{-p}(z, 0) + [1.5 + 0.5r + (0.75 - 0.75r)\alpha_2] z^2 \tag{3.4}$$

for all $z \in [0,2]_{h\mathbb{Z}}$. Choosing $p = 1$ and $h = 0.2$, one gets

$$[\Phi(z)]^r = [(0.75 + 0.25r) e_{-1}(z,0) + (1.5 + 0.5r) z^2, (1.25 - 0.25r) e_{-1}(z,0) + (2.25 - 0.25r) z^2]. \tag{3.5}$$

The graphs of (3.5) and the HMF of $\Phi(z)$ are shown in Figs. 5 and 6, respectively.

Based on the granular distance, we propose and investigate the crucial properties of the limit concept for fuzzy functions on time scales. Through this limit concept, the spaces of all continuous and right-dense continuous functions are rigorously defined.

Definition 3.2. We say that a fuzzy function $\Phi : \mathbb{T} \to \mathscr{F}_{\mathbb{R}}$ has the $\mathbb{T}$*-limit* $\Lambda \in \mathscr{F}_{\mathbb{R}}$ *at* $z_0 \in \mathbb{T}$, which we denote $\lim_{z \to z_0} \Phi(z) = \Lambda$, if for any sequence $\{z_n\}_{n \in \mathbb{N}}$ such that $z_n \neq z_0$, $z_n \in \mathbb{T}$ and $\lim_{n \to \infty} z_n = z_0$, we have $\lim_{n \to \infty} \mathfrak{D}_{gr}(\Phi(z_n), \Lambda) = 0$.

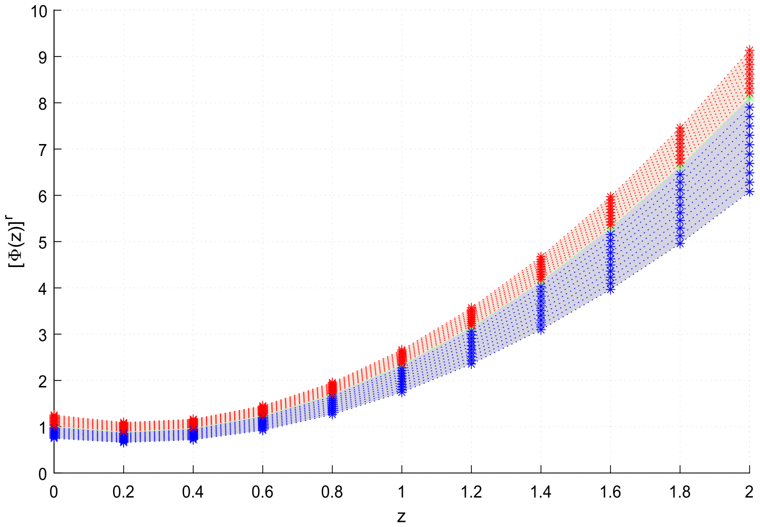

Fig. 5. The r-level set of $\Phi(z)$. The lower and upper borders of $[\Phi(z)]^r$ are depicted by the blue-dashed curves and red-dashed curves, and the green star-dashed curve is with $r = 1$.

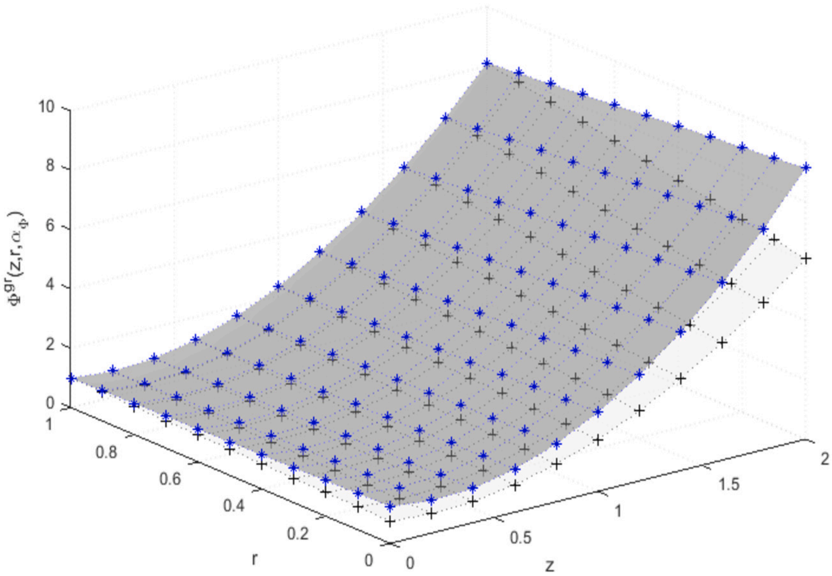

Fig. 6. The graph of $\Phi^{\mathrm{gr}}(z, r, \alpha_\Phi)$ with $\alpha_\Phi = 0$ (the black-crossed grid) and $\alpha_\Phi = 1$ (the blue-stared grid).

Definition 3.3. We say that a fuzzy function $\Phi : \mathbb{T} \to \mathscr{F}_{\mathbb{R}}$ has *the left-sided (right-sided)* $\mathbb{T}$*-limit* $\Lambda \in \mathscr{F}_{\mathbb{R}}$ *at* $z_0 \in \mathbb{T}$, which we denote $\lim_{z \to z_0^-} \Phi(z) = \Lambda$ (or $\lim_{z \to z_0^+} \Phi(z) = \Lambda$), if for any sequence $\{z_n\}_{n \in \mathbb{N}}$ such that $z_n < z_0$ (or $z_n > z_0$), $z_n \in \mathbb{T}$ and $\lim_{n \to \infty} z_n = z_0$, we have $\lim_{n \to \infty} \mathfrak{D}_{\mathrm{gr}}(\Phi(z_n), \Lambda) = 0$.

It derives from Definitions 3.2 and 3.3 that for a fuzzy function $\Phi : \mathbb{T} \to \mathscr{F}_{\mathbb{R}}$ and $\Lambda \in \mathscr{F}_{\mathbb{R}}$, $\lim_{z \to z_0} \Phi(z) = \Lambda$ iff $\lim_{z \to z_0^-} \Phi(z) = \lim_{z \to z_0^+} \Phi(z) = \Lambda$.

The remainder of this section aims to present the fundamental characteristics of limits, serving as natural extensions of the limits associated with classical real-valued functions on time scales.

Theorem 3.1. *Let* $\Phi : \mathbb{T} \to \mathscr{F}_{\mathbb{R}}$ *and* $z_0 \in \mathbb{T}$*. The* $\mathbb{T}$*-limit of* Φ *at* z_0 *(provided that it exists) is unique.*

Proof. The proof follows from Definition 3.2 and the properties of $\mathfrak{D}_{\mathrm{gr}}$. $\square$

Theorem 3.2. *Let $\Phi : \mathbb{T} \to \mathscr{F}_{\mathbb{R}}$, $z_0 \in \mathbb{T}$ and $\Lambda \in \mathscr{F}_{\mathbb{R}}$. Then, $\lim_{z\to z_0} \Phi(z) = \Lambda$ iff for every $\varepsilon > 0$, there exists a $\delta > 0$ satisfying $\mathfrak{D}_{\mathrm{gr}}(\Phi(z), \Lambda) < \varepsilon$, for all $z \in U_{\mathbb{T}}(z_0, \delta)\backslash\{z_0\}$.*

Proof. Since the sufficient condition is trivial, we will only present a proof for the necessary condition. First, we assume that $z_0 \in \mathbb{T}$ is a dense point. Assuming that $\lim_{z\to z_0} \Phi(z) = \Lambda$, we need to show that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that the following holds:

$$\mathfrak{D}_{\mathrm{gr}}(\Phi(z), \Lambda) < \varepsilon \qquad \text{for any } z \in U_{\mathbb{T}}(z_0, \delta) \setminus \{z_0\}.$$

Through a contradiction, let us assume that there exists $\varepsilon > 0$ such that for every $n \in \mathbb{N}$, there is $z_n \in U_{\mathbb{T}}(z_0, \frac{1}{n}) \setminus \{z_0\}$ and $\mathfrak{D}_{\mathrm{gr}}(\Phi(z_n), \Lambda) \ge \varepsilon$. The latter implies that $\lim_{n\to\infty} z_n = z_0$ and $\lim_{n\to\infty} \mathfrak{D}_{\mathrm{gr}}(\Phi(z_n), \Lambda) \ge \varepsilon$, provided it exists. This statement contradicts $\lim_{z\to z_0} \Phi(z) = \Lambda$. If z_0 is a scattered point, then the conclusion follows obviously, and the proof is finished. □

Proposition 3.1. *Let Φ and Ψ be fuzzy functions on $\mathbb{T}$ and let $\beta_1, \beta_2 \in \mathbb{R}$ and $z_0 \in \mathbb{T}$. If $\lim_{z\to z_0} \Phi(z) = \Lambda$ and $\lim_{z\to z_0} \Psi(z) = \tilde{\Lambda}$, where $\Lambda, \tilde{\Lambda} \in \mathscr{F}_{\mathbb{R}}$, then*

i) $\lim_{z\to z_0} \left[\beta_1\Phi(z) \oplus_{\mathrm{gr}} \beta_2\Psi(z)\right] = \beta_1\Lambda \oplus_{\mathrm{gr}} \beta_2\tilde{\Lambda}$,
ii) $\lim_{z\to z_0} \left[\beta_1\Phi(z) \ominus_{\mathrm{gr}} \beta_2\Psi(z)\right] = \beta_1\Lambda \ominus_{\mathrm{gr}} \beta_2\tilde{\Lambda}$.

Proof. We only prove part i) since part ii) follows in a similar manner. Let us assume, without loss of generality, that β_1 and β_2 are both non-zero. Since $\lim_{z\to z_0} \Phi(z) = \Lambda$ and $\lim_{z\to z_0} \Psi(z) = \tilde{\Lambda}$, according to Theorem 3.2, for any $\varepsilon > 0$, there is a $\delta > 0$ such that $\mathfrak{D}_{\mathrm{gr}}(\Phi(z), \Lambda) < \frac{\varepsilon}{2|\beta_1|}$ and $\mathfrak{D}_{\mathrm{gr}}(\Psi(z), \tilde{\Lambda}) < \frac{\varepsilon}{2|\beta_2|}$ for all $z \in U_{\mathbb{T}}(z_0, \delta) \setminus \{z_0\}$. Then, one has

$$\begin{aligned}\mathfrak{D}_{\mathrm{gr}}\left[\beta_1\Phi(z) \oplus_{\mathrm{gr}} \beta_2\Psi(z), \beta_1\Lambda \oplus_{\mathrm{gr}} \beta_2\tilde{\Lambda}\right] &\le \mathfrak{D}_{\mathrm{gr}}\left(\beta_1\Phi(z), \beta_1\Lambda\right) + \mathfrak{D}_{\mathrm{gr}}\left(\beta_2\Psi(z), \beta_2\tilde{\Lambda}\right)\\ &\le |\beta_1|\mathfrak{D}_{\mathrm{gr}}(\Phi(z), \Lambda) + |\beta_2|\mathfrak{D}_{\mathrm{gr}}(\Psi(z), \tilde{\Lambda})\\ &< |\beta_1|\frac{\varepsilon}{2|\beta_1|} + |\beta_2|\frac{\varepsilon}{2|\beta_2|} = \varepsilon,\end{aligned} \tag{3.6}$$

which implies that $\lim_{z\to z_0} \left[\beta_1\Phi(z) \oplus_{\mathrm{gr}} \beta_2\Psi(z)\right] = \beta_1\Lambda \oplus_{\mathrm{gr}} \beta_2\tilde{\Lambda}$. □

Definition 3.4.

(i) A fuzzy function $\Phi : [a, b]_{\mathbb{T}} \to \mathscr{F}_{\mathbb{R}}$ is called *continuous* at $z_0 \in [a, b]_{\mathbb{T}}$ if $\lim_{z\to z_0} \Phi(z) = \Phi(z_0)$.
(ii) A fuzzy function $\Phi : [a, b]_{\mathbb{T}} \to \mathscr{F}_{\mathbb{R}}$ is said to be continuous on $[a, b]_{\mathbb{T}}$ if it is continuous at every $z_0 \in [a, b]_{\mathbb{T}}$. Denote by $C([a, b]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}})$ the family of continuous fuzzy functions on $[a, b]_{\mathbb{T}}$.
(iii) A function Φ is considered *rd-continuous* if it exhibits continuity at right-dense points in $\mathbb{T}$, and its left-sided limits exist (and are finite) at left-dense points in $\mathbb{T}$. The collection of all rd-continuous functions $\Phi : [a, b]_{\mathbb{T}} \to \mathscr{F}_{\mathbb{R}}$ is denoted as $C_{\mathrm{rd}}([a, b]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}})$.
(iv) A fuzzy function $F : [a, b]_{\mathbb{T}} \times \mathscr{F}_{\mathbb{R}} \to \mathscr{F}_{\mathbb{R}}$ is said to be *rd-continuous* if $G(z) := F(z, \Phi(z))$ is rd-continuous for any continuous function $\Phi : [a, b]_{\mathbb{T}} \to \mathscr{F}_{\mathbb{R}}$.

Now, we are in a position to extend the concepts of delta differentiability and delta integrability to fuzzy functions on time scales.

Definition 3.5. Let $\mathbb{T}$ be an arbitrary time scale and $z_0 \in \mathbb{T}^{\kappa}$, and let $\Phi : \mathbb{T} \to \mathscr{F}_{\mathbb{R}}$ be a fuzzy function. The *granular delta derivative* (Δ^{gr}*-derivative*, for short) of $\Phi(z)$ at z_0, provided that it exists, is a fuzzy number denoted by $\Phi^{\Delta^{\mathrm{gr}}}(z_0)$, satisfying the property that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\mathfrak{D}_{\mathrm{gr}}\left(\Phi(\sigma(z_0)) \ominus_{\mathrm{gr}} \Phi(z), \Phi^{\Delta^{\mathrm{gr}}}(z_0)(\sigma(z_0) - z)\right) \le \varepsilon|\sigma(z_0) - z| \tag{3.7}$$

for all $z \in U_{\mathbb{T}}(z_0, \delta)$. The function Φ is called *granular delta differentiable* (Δ^{gr}*-differentiable* at $z_0 \in \mathbb{T}^{\kappa}$ if Φ has the granular delta derivative at z_0. Furthermore, we say that Φ is *granular delta differentiable* (Δ^{gr}*-differentiable*, for short) on $\mathbb{T}^{\kappa}$ if $\Phi^{\Delta^{\mathrm{gr}}}(z_0)$ exists for every $z_0 \in \mathbb{T}^{\kappa}$.

The next theorem, proven in [7], provides the sufficient condition for the Δ^{gr}-differentiability of fuzzy functions on time scales. It also determines the link between the existence of a fuzzy function's granular delta derivative and that of its corresponding HMF.

Theorem 3.3. *Let $\mathbb{T}$ be an arbitrary time scale and let $z_0 \in \mathbb{T}^{\kappa}$. Then, a fuzzy function $\Phi : \mathbb{T} \to \mathscr{F}_{\mathbb{R}}$ is Δ^{gr}-differentiable at z_0 if and only if its HMF, $\mathcal{H}(\Phi(z))$, is Δ-differentiable w.r.t. z at z_0. In addition,*

$$\mathcal{H}\left(\Phi^{\Delta^{\mathrm{gr}}}(z_0)\right)=\frac{\partial\Phi^{\mathrm{gr}}(z_0,r,\alpha_\Phi)}{\Delta z}, \tag{3.8}$$

where $\frac{\partial\Phi^{\mathrm{gr}}(z_0,r,\alpha_\Phi)}{\Delta z}$ *is the delta partial derivative of* Φ^{gr} *w.r.t.* z *at* z_0.

Theorem 3.4. *Let* $\mathbb{T}$ *be an arbitrary time scale,* $z_0\in\mathbb{T}^\kappa$, *and let* $\Phi:\mathbb{T}\to\mathscr{F}_{\mathbb{R}}$ *be a fuzzy function. Then, the assertions below hold:*

i) *If* Φ *is* Δ^{gr}*-differentiable at* z_0, *then* Φ *is continuous at* z_0.
ii) *If* z_0 *is a right-scattered point and* Φ *is continuous at* z_0, *then* Φ *is* Δ^{gr}*-differentiable at* z_0 *and*

$$\Phi^{\Delta^{\mathrm{gr}}}(z_0)=\frac{\Phi(\sigma(z_0))\ominus_{\mathrm{gr}}\Phi(z_0)}{\mu(z_0)}. \tag{3.9}$$

iii) *If* z_0 *is a right-dense point, then* Φ *is* Δ^{gr}*-differentiable at* z_0 *iff*

$$\lim_{z\to z_0}\frac{\Phi(z_0)\ominus_{\mathrm{gr}}\Phi(z)}{z_0-z}$$

exists. Moreover,

$$\Phi^{\Delta^{\mathrm{gr}}}(z_0)=\lim_{z\to z_0}\frac{\Phi(z_0)\ominus_{\mathrm{gr}}\Phi(z)}{z_0-z}. \tag{3.10}$$

iv) *If* Φ *is* Δ^{gr}*-differentiable at* z_0, *then*

$$\Phi(\sigma(z_0))=\Phi(z_0)\oplus_{\mathrm{gr}}\mu(z_0)\Phi^{\Delta^{\mathrm{gr}}}(z_0). \tag{3.11}$$

Proof. i) Assume that Φ is Δ^{gr}-differentiable at z_0, and let $\Phi^{\mathrm{gr}}(\cdot,r,\alpha_\Phi)$ represent its HMF. Choosing an arbitrary $\varepsilon\in(0,1)$, we define $\bar\varepsilon=\frac{\varepsilon}{1+\overline{m}+2\mu(z_0)}$, where $\overline{m}=\sup_r\max_{\alpha_\Phi}\left|\frac{\partial\Phi^{\mathrm{gr}}(z_0,r,\alpha_\Phi)}{\Delta z}\right|$. By Definition 3.5, there exists a $\delta=\bar\varepsilon>0$ such that the following holds:

$$\mathfrak{D}_{\mathrm{gr}}\left(\Phi(\sigma(z_0))\ominus_{\mathrm{gr}}\Phi(z),\Phi^{\Delta^{\mathrm{gr}}}(z_0)(\sigma(z_0)-z)\right)\le\bar\varepsilon|\sigma(z_0)-z| \tag{3.12}$$

for all $z\in U_{\mathbb{T}}(z_0,\delta)$, which yields

$$\sup_r\max_{\alpha_\Phi}|\mathbb{P}_z|\le\bar\varepsilon|\sigma(z_0)-z|, \tag{3.13}$$

where $\mathbb{P}_z=\Phi^{\mathrm{gr}}(\sigma(z_0),r,\alpha_\Phi)-\Phi^{\mathrm{gr}}(z,r,\alpha_\Phi)-\frac{\partial\Phi^{\mathrm{gr}}(z_0,r,\alpha_\Phi)}{\Delta z}(\sigma(z_0)-z)$ for all $r,\alpha_\Phi\in[0,1]$. Moreover, one has the estimation

$$\begin{aligned}
&\left|\Phi^{\mathrm{gr}}(z_0,r,\alpha_\Phi)-\Phi^{\mathrm{gr}}(z,r,\alpha_\Phi)\right|\\
&=\Bigg|\Phi^{\mathrm{gr}}(z_0,r,\alpha_\Phi)-\Phi^{\mathrm{gr}}(z,r,\alpha_\Phi)+\Phi^{\mathrm{gr}}(\sigma(z_0),r,\alpha_\Phi)-\frac{\partial\Phi^{\mathrm{gr}}(z_0,r,\alpha_\Phi)}{\Delta z}(\sigma(z_0)-z)\\
&\qquad+\frac{\partial\Phi^{\mathrm{gr}}(z_0,r,\alpha_\Phi)}{\Delta z}(\sigma(z_0)-z)-\Phi^{\mathrm{gr}}(\sigma(z_0),r,\alpha_\Phi)\Bigg|\\
&\le|\mathbb{P}_z|+\left|\Phi^{\mathrm{gr}}(\sigma(z_0),r,\alpha_\Phi)-\Phi^{\mathrm{gr}}(z_0,r,\alpha_\Phi)-\frac{\partial\Phi^{\mathrm{gr}}(z_0,r,\alpha_\Phi)}{\Delta z}(\sigma(z_0)-z)\right|\\
&\le|\mathbb{P}_z|+|\mathbb{P}_{z_0}|+\overline{m}|z-z_0|.
\end{aligned} \tag{3.14}$$

Combining (3.13) and (3.14) with the property of the supremum, we obtain

$$\begin{aligned}
\sup_r\max_{\alpha_\Phi}\left|\Phi^{\mathrm{gr}}(z_0,r,\alpha_\Phi)-\Phi^{\mathrm{gr}}(z,r,\alpha_\Phi)\right|&\le\sup_r\max_{\alpha_\Phi}|\mathbb{P}_z|+\sup_r\max_{\alpha_\Phi}|\mathbb{P}_{z_0}|+\overline{m}|z-z_0|\\
&\le\bar\varepsilon|\sigma(z_0)-z|+\bar\varepsilon|\sigma(z_0)-z_0|+\overline{m}|z-z_0|\\
&\le\bar\varepsilon(|\sigma(z_0)-z|+\mu(z_0)+\overline{m}).
\end{aligned} \tag{3.15}$$

Therefore, we get $\mathfrak{D}_{\mathrm{gr}}\left(\Phi(z_0),\Phi(z)\right)\le\bar\varepsilon(|z_0-z|+2\mu(z_0)+\overline{m})\le\bar\varepsilon(1+\overline{m}+2\mu(z_0))=\varepsilon$, which yields that Φ is continuous at z_0.
ii) Since Φ is continuous at z_0, it follows from Proposition 3.1 that

$$\lim_{z\to z_0}\frac{\Phi(\sigma(z_0))\ominus_{\mathrm{gr}}\Phi(z)}{\sigma(z_0)-z}=\frac{\Phi(\sigma(z_0))\ominus_{\mathrm{gr}}\Phi(z_0)}{\sigma(z_0)-z_0}=\frac{\Phi(\sigma(z_0))\ominus_{\mathrm{gr}}\Phi(z_0)}{\mu(z_0)}. \tag{3.16}$$

Combining (3.16) with Theorem 3.2, one receives that for an arbitrary $\varepsilon>0$, there is a number $\delta>0$ satisfying

$$\mathfrak{D}_{\mathrm{gr}}\left(\frac{\Phi(\sigma(z_0))\ominus_{\mathrm{gr}}\Phi(z)}{\sigma(z_0)-z},\frac{\Phi(\sigma(z_0))\ominus_{\mathrm{gr}}\Phi(z_0)}{\mu(z_0)}\right)<\varepsilon \tag{3.17}$$

for all $z \in U_{\mathbb{T}}(z_0, \delta) \setminus \{z_0\}$, which is written by

$$\mathfrak{D}_{\mathrm{gr}}\left(\Phi(\sigma(z_0)) \ominus_{\mathrm{gr}} \Phi(z), \frac{\Phi(\sigma(z_0)) \ominus_{\mathrm{gr}} \Phi(z_0)}{\mu(z_0)}(\sigma(z_0) - z)\right) < \varepsilon|\sigma(z_0) - z|. \tag{3.18}$$

It means that Φ is Δ^{gr}-differentiable at z_0 and (3.9) holds.
iii) Assume that z_0 is a right-dense point and Φ is Δ^{gr}-differentiable at z_0. Then, for any $\varepsilon > 0$, there exists a $\delta > 0$ satisfying

$$\mathfrak{D}_{\mathrm{gr}}\left(\frac{\Phi(z_0) \ominus_{\mathrm{gr}} \Phi(z)}{z_0 - z}, \Phi^{\Delta^{\mathrm{gr}}}(z_0)\right) < \varepsilon \text{ for all } z \in U_{\mathbb{T}}(z_0, \delta) \setminus \{z_0\}. \tag{3.19}$$

Thus, by Theorem 3.2, $\lim\limits_{z\to z_0} \frac{\Phi(z_0) \ominus_{\mathrm{gr}} \Phi(z)}{z_0 - z}$ exists and $\Phi^{\Delta^{\mathrm{gr}}}(z_0) = \lim\limits_{z\to z_0} \frac{\Phi(z_0) \ominus_{\mathrm{gr}} \Phi(z)}{z_0 - z}$.
iv) If z_0 is right-dense, the above assertion is obvious. Let z_0 be right-scattered and let Φ be Δ^{gr}-differentiable at z_0. Then, it yields from i) and ii) that

$$\mu(z_0)\Phi^{\Delta^{\mathrm{gr}}}(z_0) = \Phi(\sigma(z_0)) \ominus_{\mathrm{gr}} \Phi(z_0). \tag{3.20}$$

Consequently, one implies $\mu(z_0)\mathcal{H}\left(\Phi^{\Delta^{\mathrm{gr}}}(z_0)\right) = \mathcal{H}\left(\Phi(\sigma(z_0))\right) - \mathcal{H}\left(\Phi(z_0)\right)$. It follows that

$$\mathcal{H}\left(\Phi(\sigma(z_0))\right) = \mathcal{H}\left(\Phi(z_0) \oplus_{\mathrm{gr}} \mu(z_0)\Phi^{\Delta^{\mathrm{gr}}}(z_0)\right). \quad \square$$

Remark 3.1. We examine several special cases involving well-known time scales.
Case 1: Let $\mathbb{T} = \mathbb{R}$. Then, every $z_0 \in \mathbb{T}$ is dense. Thus, by iii) of Theorem 3.4 for $\Phi : [a, b] \to \mathscr{F}_{\mathbb{R}}$ and $z_0 \in [a, b]$, one gets

$$\Phi^{\Delta^{\mathrm{gr}}}(z_0) = \lim_{\ell\to 0} \frac{\Phi(z_0 + \ell) \ominus_{\mathrm{gr}} \Phi(z_0)}{\ell}. \tag{3.21}$$

In this situation, $\Phi^{\Delta^{\mathrm{gr}}}(z_0)$ is said to be the *granular derivative* of the fuzzy function Φ at z_0, as introduced in [27].

Case 2: For a given $h > 0$, let us consider $\mathbb{T} = h\mathbb{Z}$. Then, by Theorem 3.4-ii), we conclude that $\Phi : [a, b]_{\mathbb{T}} \to \mathscr{F}_{\mathbb{R}}$ is Δ^{gr}-differentiable on $[a, b]_{\mathbb{T}}$, and

$$\Delta_h^{\mathrm{gr}}\Phi(z_0) = \frac{1}{h}\left[\Phi(z_0 + h) \ominus_{\mathrm{gr}} \Phi(z_0)\right] \tag{3.22}$$

for all $z_0 \in [a, b]_{\mathbb{T}}$. In this case, Δ_h^{gr} is said to be the *granular h-difference operator*.
Case 3: Let $\mathbb{T} = \mathbb{T}_q = \{q^n \mid n \in \mathbb{N}_0\} \cup \{0\}$, $q \in (0, 1)$. Then, by Theorem 3.4, the granular delta derivative of $\Phi : \mathbb{T}_q \to \mathscr{F}_{\mathbb{R}}$ is

$$D_q^{\mathrm{gr}}\Phi(z_0) = \frac{\Phi(q^{-1}z_0) \ominus_{\mathrm{gr}} \Phi(z_0)}{(q^{-1} - 1)z_0} \tag{3.23}$$

for every $z_0 \in \mathbb{T}_q \backslash \{0\}$. For $z_0 = 0$, we have $D_q^{\mathrm{gr}}\Phi(0) = \lim\limits_{n\to\infty} \frac{\Phi(q^n) \ominus_{\mathrm{gr}} \Phi(0)}{q^n}$. Accordingly, D_q^{gr} is called the *granular q-derivative* (see [44]).

Example 3.2. Let us consider the fuzzy function Φ as given in Example 3.1, namely

$$\Phi(z) = \tilde{1}e_{-p}(z, 0) \oplus_{\mathrm{gr}} \tilde{2}z^2, \tag{3.24}$$

where $\tilde{1} = (0.75, 1, 1.25)$, $\tilde{2} = (1.5, 2, 2.25) \in \mathscr{F}_{\mathbb{R}}$, and $-p \in \mathcal{R}(\mathbb{T}, \mathbb{R})$. Observe that

$$\Phi^{\mathrm{gr}}(z, r, \alpha_\Phi) = [0.75 + 0.25r + (0.5 - 0.5r)\alpha_1]e_{-p}(z, 0) + [1.5 + 0.5r + (0.75 - 0.75r)\alpha_2]z^2 \tag{3.25}$$

for all $z \in [0, 2]_{h\mathbb{Z}}, r, \alpha_1, \alpha_2 \in [0, 1]$. By Theorem 3.3,

$$\mathcal{H}\left(\Phi^{\Delta^{\mathrm{gr}}}(z)\right) = -p[0.75 + 0.25r + (0.5 - 0.5r)\alpha_1]e_{-p}(z, 0) + [1.5 + 0.5r + (0.75 - 0.75r)\alpha_2](2z + h) \tag{3.26}$$

for all $z \in [0, 2)_{h\mathbb{Z}}, r, \alpha_1, \alpha_2 \in [0, 1]$. If $h = 0.2$ and $p = 1$, then

$$\mathcal{H}\left(\Phi^{\Delta^{\mathrm{gr}}}(z)\right) = [-0.75 - 0.25r + (0.5r - 0,5)\alpha_1]e_{-1}(z, 0) + [1.5 + 0.5r + (0.75 - 0.75r)\alpha_2](2z + 0.2). \tag{3.27}$$

The graph of the HMF of $\Phi^{\Delta^{\mathrm{gr}}}(z)$ is shown in Fig. 8. Therefore, by formula (2.6), one obtains

$$[\Phi^{\Delta^{\mathrm{gr}}}(z)]^r = [(-1.25 + 0.25r)e_{-1}(z, 0) + (2z + 0.2)(1.5 + 0.5r), (-0.75 - 0.25r)e_{-1}(z, 0) + (2z + 0.2)(2.25 - 0.25r)] \tag{3.28}$$

for all $z \in [0, 2)_{h\mathbb{Z}}$. The graph of $[\Phi^{\Delta^{\mathrm{gr}}}(z)]^r$ is shown in Fig. 7.

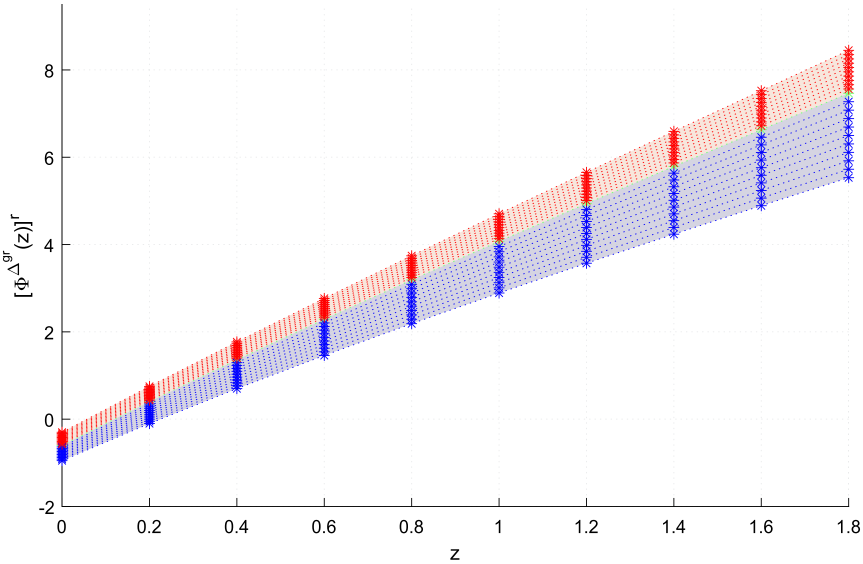

Fig. 7. The r-level set of $\Phi^{\Delta^{gr}}(z)$. The lower and upper borders of $[\Phi^{\Delta^{gr}}(z)]^r, z \in [0,2)_{h\mathbb{Z}}$, are depicted by the blue-dashed curves and red-dashed curves, and the green star-dashed curve is for $r = 1$.

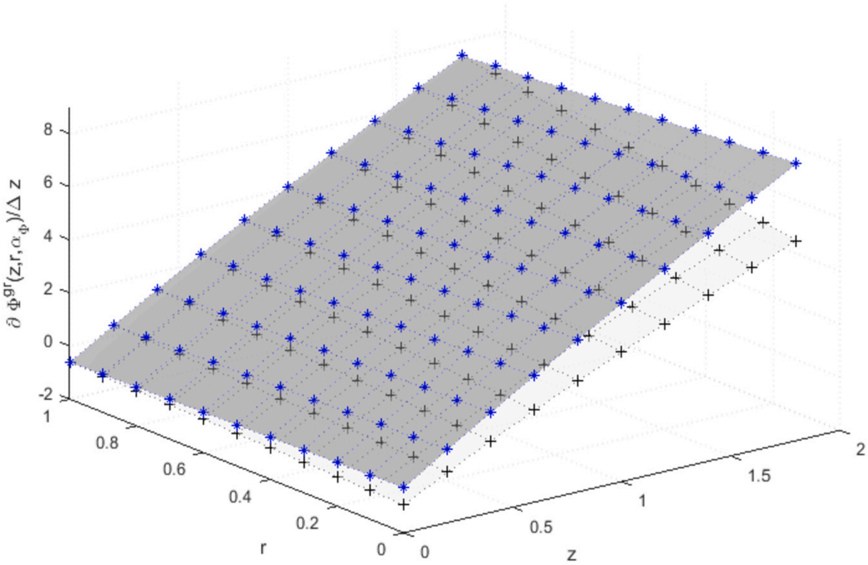

Fig. 8. The graph of the HMF of $\Phi^{\Delta^{gr}}(z)$ with $\alpha_\Phi = 0$ (the black-crossed grid) and $\alpha_\Phi = 1$ (the blue-stared grid).

Remark 3.2. According to Theorem 3.3, the properties of Δ^{gr}-differentiability of fuzzy functions are entirely inherited from the differentiability of real functions on time scales, presented in Theorem 2.1.

Based on the HMF approach for fuzzy numbers, the concept of the granular delta integral for fuzzy functions on time scales is defined as follows

Definition 3.6. Let $a, b \in \mathbb{T}$ and $a < b$. Let $\Phi : \mathbb{T} \to \mathcal{F}_{\mathbb{R}}$ be a fuzzy function with its HMF $\Phi^{gr}(\cdot, r, \alpha_\Phi) \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ for all $r, \alpha_\Phi \in [0,1]$. We say that the function Φ is *granular delta integrable* on $[a,b]_{\mathbb{T}}$ if there is a fuzzy number N such that

$$\mathcal{H}(N) = \int_a^b \Phi^{gr}(z, r, \alpha_\Phi)\Delta z. \tag{3.29}$$

In this case, N is said to be the *granular delta integral* of Φ from a to b, denoted by $\int_a^b \Phi(z)\Delta z$.

The following theorem presents some essential properties of the granular delta integral of fuzzy functions on time scales.

Theorem 3.5. *Let $a, b \in \mathbb{T}$ and $a < b$. Assume that $\Psi, \Phi : \mathbb{T} \to \mathscr{F}_{\mathbb{R}}$ are granular delta integrable. The following assertions hold:*

i) $\int_a^b [\beta_1 \Psi(z) \circledast_{gr} \beta_2 \Phi(z)]\Delta z = \beta_1 \int_a^b \Psi(z)\Delta z \circledast_{gr} \beta_2 \int_a^b \Phi(z)\Delta z$, *where $\circledast_{gr}$ represents $\oplus_{gr}$ or $\ominus_{gr}$ and $\beta_1, \beta_2 \in \mathbb{R}$,*

ii) $\mathfrak{D}_{gr}\left(\int_a^b \Psi(z)\Delta z, \int_a^b \Phi(z)\Delta z\right) \le \int_a^b \mathfrak{D}_{gr}(\Psi(z), \Phi(z))\Delta z.$

Proof. The property i) is directly obtained from Definition 3.6 and well-known properties of the delta integral on time scales, thus the proof is omitted. To verify ii), we assume that $\Psi, \Phi : \mathbb{T} \to \mathscr{F}_{\mathbb{R}}$ are granular delta integrable. Let $\Psi^{gr}(z, r, \alpha_\Psi)$ and $\Phi^{gr}(z, r, \alpha_\Phi)$ be HMFs of $\Psi(z)$ and $\Phi(z)$, respectively. Observe that

$$\left|\int_a^b \Psi^{gr}(z, r, \alpha_\Psi)\Delta z - \int_a^b \Phi^{gr}(z, r, \alpha_\Phi)\Delta z\right| \le \int_a^b \left|\Psi^{gr}(z, r, \alpha_\Psi) - \Phi^{gr}(z, r, \alpha_\Phi)\right| \Delta z \tag{3.30}$$

for all $r, \alpha_\Psi, \alpha_\Phi \in [0, 1]$, which yields that

$$\sup_{r\in[0,1]} \max_{\alpha_\Psi,\alpha_\Phi\in[0,1]} \left|\int_a^b \Psi^{gr}(z, r, \alpha_\Psi)\Delta z - \int_a^b \Phi^{gr}(z, r, \alpha_\Phi)\Delta z\right| \le \sup_{r\in[0,1]} \max_{\alpha_\Psi,\alpha_\Phi\in[0,1]} \int_a^b \left|\Psi^{gr}(z, r, \alpha_\Psi) - \Phi^{gr}(z, r, \alpha_\Phi)\right| \Delta z. \tag{3.31}$$

Hence,

$$\begin{aligned}
\mathfrak{D}_{gr}\left(\int_a^b \Psi(z)\Delta z, \int_a^b \Phi(z)\Delta z\right) &= \sup_{r\in[0,1]} \max_{\alpha_\Psi,\alpha_\Phi\in[0,1]} \left|\int_a^b \Psi^{gr}(z, r, \alpha_\Psi)\Delta z - \int_a^b \Phi^{gr}(z, r, \alpha_\Phi)\Delta z\right| \\
&\le \int_a^b \sup_{r\in[0,1]} \max_{\alpha_\Psi,\alpha_\Phi\in[0,1]} \left|\Psi^{gr}(z, r, \alpha_\Psi) - \Phi^{gr}(z, r, \alpha_\Phi)\right| \Delta z \\
&= \int_a^b \mathfrak{D}_{gr}(\Psi(z), \Phi(z))\Delta z. \quad \square
\end{aligned} \tag{3.32}$$

The next theorem is the generalization of the Newton–Leibniz theorem.

Theorem 3.6. *Let $\Phi : \mathbb{T} \to \mathscr{F}_{\mathbb{R}}$ be granular delta differentiable, and $a, b \in \mathbb{T}$. If $F(z) = \Phi^{\Delta^{gr}}(z)$ is rd-continuous for $z \in \mathbb{T}^\kappa$, then*

$$\int_a^b F(z)\Delta z = \Phi(b) \ominus_{gr} \Phi(a). \tag{3.33}$$

Proof. Since Φ is Δ^{gr}-differentiable, one receives from Theorem 3.3 that

$$\mathcal{H}\left(\Phi^{\Delta^{gr}}(z)\right) = \frac{\partial \Phi^{gr}(z, r, \alpha_\Phi)}{\Delta z} \quad \text{for all } r, \alpha_\Phi \in [0, 1].$$

Using Definition 2.2, one has

$$\int_a^b \frac{\partial \Phi^{gr}(z, r, \alpha_\Phi)}{\Delta z}\Delta z = \Phi^{gr}(b, r, \alpha_\Phi) - \Phi^{gr}(a, r, \alpha_\Phi) \quad \text{for all } r, \alpha_\Phi \in [0, 1],$$

i.e.,

$$\int_a^b \mathcal{H}(F(z))\,\Delta z = \Phi^{\mathrm{gr}}(b,r,\alpha_\Phi) - \Phi^{\mathrm{gr}}(a,r,\alpha_\Phi) \quad \text{for all } r,\alpha_\Phi \in [0,1].$$

In addition, it follows from Definition 2.4 that

$$\mathcal{H}\left(\Phi(b)\ominus_{\mathrm{gr}}\Phi(a)\right) = \Phi^{\mathrm{gr}}(b,r,\alpha_\Phi) - \Phi^{\mathrm{gr}}(a,r,\alpha_\Phi) \quad \text{for all } r,\alpha_\Phi \in [0,1],$$

and hence $\int_a^b \mathcal{H}(F(z))\,\Delta z = \mathcal{H}\left(\Phi(b)\ominus_{\mathrm{gr}}\Phi(a)\right)$. □

Example 3.3. Let us reconsider the fuzzy function $\Phi : [0,2]_{h\mathbb{Z}} \to \mathscr{F}_\mathbb{R}$ as given in Example 3.1 with $0<h<1$. Set $G(z)=\Phi^{\Delta^{\mathrm{gr}}}(z)$ for all $z\in[0,2)_{h\mathbb{Z}}$. According to Example 3.2, one has

$$\mathcal{H}(G(z)) = \mathcal{H}(\Phi^{\Delta^{\mathrm{gr}}}(z)) = -p[0.75+0.25r+(0.5-0.5r)\alpha_1]e_{-p}(z,0) + [1.5+0.5r+(0.75-0.75r)\alpha_2](2z+h)$$

for all $\alpha_1,\alpha_2\in[0,1]$, $z\in[-2,2)_{h\mathbb{Z}}$. By Theorem 3.6, one has $\int_0^1 G(z)\Delta z = \Phi(1)\ominus_{\mathrm{gr}}\Phi(0)$. Indeed, using Definition 2.4, one derives

$$\mathcal{H}\left(\Phi(1)\ominus_{\mathrm{gr}}\Phi(0)\right) = [0.75+0.25r+(0.5-0.5r)\alpha_1]e_{-p}(1,0) + 1.5+0.5r+(0.75-0.75r)\alpha_2.$$

From Definition 3.6, we have

$$\begin{aligned}\mathcal{H}\left(\int_0^1 G(z)\Delta z\right) &= \int_0^1 \mathcal{H}(G(z))\Delta z \\ &= -p[0.75+0.25r+(0.5-0.5r)\alpha_1]h\sum_{k=0}^{\frac{1}{h}-1}(1-ph)^{\frac{kh}{h}} + [1.5+0.5r+(0.75-0.75r)\alpha_2]h\sum_{k=0}^{\frac{1}{h}-1}(2kh+h) \\ &= [0.75+0.25r+(0.5-0.5r)\alpha_1]e_{-p}(1,0)+1.5+0.5r+(0.75-0.75r)\alpha_2, \end{aligned} \tag{3.34}$$

which implies that $\mathcal{H}\left(\int_0^1 G(z)\Delta z\right) = \mathcal{H}\left(\Phi(1)\ominus_{\mathrm{gr}}\Phi(0)\right)$. Therefore, it follows from Definition 2.5 that $\int_0^1 G(z)\Delta z = \Phi(1)\ominus_{\mathrm{gr}}\Phi(0)$.

4. Fuzzy dynamic equations on time scales

We consider the initial value problem of fuzzy dynamic equations on an arbitrary time scale $\mathbb{T}$ in the following form:

$$\begin{cases} Y^{\Delta^{\mathrm{gr}}}(z) = F(z,Y(z)) & \text{for all } z\in[z_0,z_0+\lambda]_\mathbb{T}^\kappa, \\ Y(z_0)=Y_0\in\mathscr{F}_\mathbb{R}, \end{cases} \tag{4.1}$$

where $z_0,\lambda\in\mathbb{T}$, $\lambda>z_0$ is such that $z_0+\lambda\in\mathbb{T}$; $F:[z_0,z_0+\lambda]_\mathbb{T}^\kappa\times\mathscr{F}_\mathbb{R}\to\mathscr{F}_\mathbb{R}$ is a fuzzy function.

Let $Y:[z_0,z_0+\lambda]_\mathbb{T}\to\mathscr{F}_\mathbb{R}$ be a fuzzy function. Then, the function Y is a solution to (4.1) if it is a Δ^{gr}-differentiable function on $[z_0,z_0+\lambda]_\mathbb{T}^\kappa$ and satisfies (4.1).

Lemma 4.1. *Let $F:[z_0,z_0+\lambda]_\mathbb{T}^\kappa\times\mathscr{F}_\mathbb{R}\to\mathscr{F}_\mathbb{R}$ be right-dense continuous. Then, Y is a solution to problem (4.1) iff Y satisfies*

$$Y(z) = Y_0 \oplus_{\mathrm{gr}} \int_{z_0}^{z} F(s,Y(s))\,\Delta s \quad \text{for all } z\in[z_0,z_0+\lambda]_\mathbb{T}. \tag{4.2}$$

Proof. Based on Definition 2.5 and Theorem 3.3, we can rewrite the initial value problem (4.1) as

$$\begin{cases} \mathcal{H}\left(Y^{\Delta^{\mathrm{gr}}}(z)\right) = \mathcal{H}(F(z,\mathcal{H}(Y(z)))) & \text{for all } z\in[z_0,z_0+\lambda]_\mathbb{T}^\kappa, \\ \mathcal{H}\left(Y(z_0)\right) = \mathcal{H}\left(Y_0\right)\in\mathscr{F}_\mathbb{R}, \end{cases} \tag{4.3}$$

i.e.,

$$\begin{cases} \frac{\partial Y^{\mathrm{gr}}(z,r,\alpha_Y)}{\Delta z} = F^{\mathrm{gr}}\left(z, Y^{\mathrm{gr}}(z,r,\alpha_Y), r, \alpha_F\right), \\ Y^{\mathrm{gr}}(z_0,r,\alpha_Y) = Y_0^{\mathrm{gr}}(r,\alpha_Y), \end{cases} \tag{4.4}$$

for all $z \in [z_0, z_0 + \lambda]_{\mathbb{T}}^{\kappa}$ and $r, \alpha_F, \alpha_Y \in [0,1]$. By taking the delta integral on both sides of the first equation in (4.4), one gets

$$\int_{z_0}^{z} \frac{\partial Y^{\mathrm{gr}}(s,r,\alpha_Y)}{\Delta s} \Delta s = \int_{z_0}^{z} F^{\mathrm{gr}}\left(s, Y^{\mathrm{gr}}(s,r,\alpha_Y), r, \alpha_F\right) \Delta s \quad \text{for all } z \in [z_0, z_0+\lambda]_{\mathbb{T}}. \tag{4.5}$$

Then, using the initial condition, we obtain

$$Y^{\mathrm{gr}}(z,r,\alpha_Y) = Y^{\mathrm{gr}}(z_0,r,\alpha_Y) + \int_{z_0}^{z} F^{\mathrm{gr}}\left(s, Y^{\mathrm{gr}}(s,r,\alpha_Y), r, \alpha_F\right) \Delta s \tag{4.6}$$

for all $z \in [z_0, z_0+\lambda]_{\mathbb{T}}, \alpha_Y, \alpha_F \in [0,1]$, which means

$$\mathcal{H}(Y(z)) = \mathcal{H}\left(Y_0\right) + \mathcal{H}\left(\int_{z_0}^{z} F(s, Y(s)) \Delta s\right). \tag{4.7}$$

Finally, one can deduce

$$Y(z) = Y_0 \oplus_{\mathrm{gr}} \int_{z_0}^{z} F(s, Y(s)) \Delta s \quad \text{for all } z \in [z_0, z_0+\lambda]_{\mathbb{T}}.$$

The reverse proof is trivial; thus, the proof is complete. □

Let $p > 0$ be a real constant. Let us consider the Bielecki metric as follows:

$$\mathfrak{D}_{\mathrm{gr}}^{p}(\Psi, \Phi) = \sup_{z \in [z_0, z_0+\lambda]_{\mathbb{T}}} \frac{\mathfrak{D}_{\mathrm{gr}}(\Psi(z), \Phi(z))}{e_p(z, z_0)} \quad \text{for all } \Psi, \Phi \in C([z_0, z_0+\lambda]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}}). \tag{4.8}$$

Lemma 4.2. *The space* $(C([z_0, z_0+\lambda]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}}), \mathfrak{D}_{\mathrm{gr}}^{p})$ *is a complete metric space.*

Proof. First, we will verify that $\mathfrak{D}_{\mathrm{gr}}^{p}$ is a metric. Indeed, for any $\Psi, \Phi \in C([z_0, z_0+\lambda]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}})$, we have $\mathfrak{D}_{\mathrm{gr}}^{p}(\Psi, \Phi) \geq 0$ because $e_p(z, z_0) > 0$ by assumption. In addition, one observes that $\sup_{z \in [z_0, z_0+\lambda]_{\mathbb{T}}} \frac{\mathfrak{D}_{\mathrm{gr}}(\Psi(z), \Phi(z))}{e_p(z, z_0)} = 0$ iff $\mathfrak{D}_{\mathrm{gr}}(\Psi(z), \Phi(z)) = 0$, which implies that $\mathfrak{D}_{\mathrm{gr}}^{p}(\Psi(z), \Phi(z)) = 0$ iff $\Psi(z) = \Phi(z)$ for all $z \in [z_0, z_0+\lambda]_{\mathbb{T}}$. Next, one has

$$\mathfrak{D}_{\mathrm{gr}}^{p}(\Psi, \Phi) = \sup_{z \in [z_0, z_0+\lambda]_{\mathbb{T}}} \frac{\mathfrak{D}_{\mathrm{gr}}(\Psi(z), \Phi(z))}{e_p(z, z_0)} = \sup_{z \in [z_0, z_0+\lambda]_{\mathbb{T}}} \frac{\mathfrak{D}_{\mathrm{gr}}(\Phi(z), \Psi(z))}{e_p(z, z_0)} = \mathfrak{D}_{\mathrm{gr}}^{p}(\Phi, \Psi)$$

for any $\Psi, \Phi \in C([z_0, z_0+\lambda]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}})$. Finally, we will check the triangular inequality. For any $\Psi, \Phi, \Theta \in C([z_0, z_0+\lambda]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}})$, we have

$$\begin{aligned}
\mathfrak{D}_{\mathrm{gr}}^{p}(\Psi, \Phi) &= \sup_{z \in [z_0, z_0+\lambda]_{\mathbb{T}}} \frac{\mathfrak{D}_{\mathrm{gr}}(\Psi(z), \Phi(z))}{e_p(z, z_0)} \\
&\leq \sup_{z \in [z_0, z_0+\lambda]_{\mathbb{T}}} \frac{\mathfrak{D}_{\mathrm{gr}}(\Psi(z), \Theta(z))}{e_p(z, z_0)} + \sup_{z \in [z_0, z_0+\lambda]_{\mathbb{T}}} \frac{\mathfrak{D}_{\mathrm{gr}}(\Theta(z), \Phi(z))}{e_p(z, z_0)} \\
&= \mathfrak{D}_{\mathrm{gr}}^{p}(\Psi, \Theta) + \mathfrak{D}_{\mathrm{gr}}^{p}(\Theta, \Phi).
\end{aligned} \tag{4.9}$$

Furthermore, since $e_p^{\Delta}(\cdot, z_0) = p e_p(\cdot, z_0) > 0$, $e_p(\cdot, z_0)$ is increasing. For all $z \in [z_0, z_0+\lambda]_{\mathbb{T}}$, we have $1 = e_p(z_0, z_0) \leq e_p(z, z_0) \leq e_p(z_0 + \lambda, z_0)$, which implies

$$1 \geq \frac{1}{e_p(z, z_0)} \geq \frac{1}{e_p(z_0+\lambda, z_0)}.$$

It yields that

$$\begin{aligned}
\sup_{z \in [z_0, z_0+\lambda]_{\mathbb{T}}} \mathfrak{D}_{\mathrm{gr}}(\Psi(z), \Phi(z)) &\geq \sup_{z \in [z_0, z_0+\lambda]_{\mathbb{T}}} \frac{\mathfrak{D}_{\mathrm{gr}}(\Psi(z), \Phi(z))}{e_p(z, z_0)} \\
&\geq \frac{1}{e_p(z_0+\lambda, z_0)} \sup_{z \in [z_0, z_0+\lambda]_{\mathbb{T}}} \mathfrak{D}_{\mathrm{gr}}(\Psi(z), \Phi(z)).
\end{aligned}$$

Therefore, the completeness of the space $(C([z_0, z_0+\lambda]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}}), \mathfrak{D}_{\mathrm{gr}}^{p})$ follows from the completeness of the space $(C([z_0, z_0+\lambda]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}}), \mathfrak{D}_{\mathrm{gr}}^{0})$, where as $\mathfrak{D}_{\mathrm{gr}}^{0}(\Psi(z), \Phi(z)) = \sup_{z \in [z_0, z_0+\lambda]_{\mathbb{T}}} \mathfrak{D}_{\mathrm{gr}}(\Psi(z), \Phi(z))$ for $\Psi, \Phi \in C([z_0, z_0+\lambda]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}})$. □

To investigate the existence of a unique solution to the initial value problem (4.1), the following hypotheses are considered.

(A) The fuzzy function $F : [z_0, z_0+\lambda]_{\mathbb{T}}^{\kappa} \times \mathscr{F}_{\mathbb{R}} \to \mathscr{F}_{\mathbb{R}}$ is rd-continuous.
(B) There exists $L > 0$ such that

$$\mathfrak{D}_{\text{gr}}\left(F(z,\Psi), F(z,\Phi)\right) \le L\mathfrak{D}_{\text{gr}}\left(\Psi,\Phi\right) \quad \text{for all } \Psi,\Phi \in C([z_0,z_0+\lambda]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}}).$$

(C) There exists $L > 0$ such that

$$\mathfrak{D}_{\text{gr}}\left(F(z,\Psi), \hat{0}\right) \le L[1+\mathfrak{D}_{\text{gr}}(\Psi,\hat{0})] \quad \text{for all } \Psi \in C([z_0,z_0+\lambda]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}}).$$

Let us introduce the norm $\|\cdot\|_{\text{gr}}^{p}$ as follows

$$\|\Psi\|_{\text{gr}}^{p} = \sup_{z\in[z_0,z_0+\lambda]_{\mathbb{T}}} \frac{\mathfrak{D}_{\text{gr}}(\Psi(z),\hat{0})}{e_p(z,z_0)}, \quad p > 0. \tag{4.10}$$

Theorem 4.1. *Let $F : [z_0, z_0+\lambda]_{\mathbb{T}}^{\kappa} \times \mathscr{F}_{\mathbb{R}} \to \mathscr{F}_{\mathbb{R}}$ satisfy the conditions (A) and (C). Then problem (4.1) has at least one solution.*

Proof. Let us introduce an operator $\mathcal{S} : C([z_0,z_0+\lambda]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}}) \to C([z_0,z_0+\lambda]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}})$ defined by

$$Y \mapsto \mathcal{S}[Y](z) = Y_0 \oplus_{\text{gr}} \int_{z_0}^{z} F(s,Y(s))\,\Delta s \text{ for all } z \in [z_0,z_0+\lambda]_{\mathbb{T}}. \tag{4.11}$$

Note that the space C with the norm given by (4.10) is a Banach space. Using Schaefer's theorem, we prove the operator $\mathcal{S}$ has a fixed point. The proof can be divided into several steps as follows. *Step 1*: The continuity of $\mathcal{S}$. Let $\{Y_n(\cdot) \mid n \in \mathbb{N}\}$ be a sequence satisfying that $Y_n \to Y$ in $C([z_0,z_0+\lambda]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}})$. Then, for each $z \in C([z_0,z_0+\lambda]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}})$, we have

$$\begin{aligned}
\mathfrak{D}_{\text{gr}}\left(\mathcal{S}[Y_n](z), \mathcal{S}[Y](z)\right) &\le \mathfrak{D}_{\text{gr}}\left(\int_{z_0}^{z} F\left(s,Y_n(s)\right)\Delta s, \int_{z_0}^{z} F\left(s,Y(s)\right)\Delta s\right) \\
&\le \int_{z_0}^{z} \mathfrak{D}_{\text{gr}}\left(F\left(s,Y_n(s)\right), F\left(s,Y(s)\right)\right)\Delta s \\
&= \int_{z_0}^{z} \sup_{r}\max_{\alpha_Y,\alpha_F}\left|F^{\text{gr}}\left(s,Y_n^{\text{gr}}(s,r,\alpha_Y),r,\alpha_F\right) - F^{\text{gr}}\left(s,Y^{\text{gr}}(s,r,\alpha_Y),r,\alpha_F\right)\right|\Delta s \to 0,
\end{aligned} \tag{4.12}$$

as $n \to \infty$, which implies the continuity of F. *Step 2*: The map $\mathcal{S}$ maps bounded sets into bounded sets in $C([z_0,z_0+\lambda]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}})$. It means that we need to show that for any $\varrho > 0$ if there exists a positive constant ρ such that for each $Y \in \mathcal{B}_{\varrho} := \{Y \in C([z_0,z_0+\lambda]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}}) \mid \mathfrak{D}_{\text{gr}}(Y,\hat{0}) \le \varrho\}$, then we obtain $\left\|\mathcal{S}[Y](z)\right\|_{\text{gr}}^{p} \le \rho$ for each $z \in [z_0,z_0+\lambda]_{\mathbb{T}}$. Indeed, from the hypotheses, one gets

$$\begin{aligned}
\mathfrak{D}_{\text{gr}}\left(\mathcal{S}[Y](z),\hat{0}\right) &= \mathfrak{D}_{\text{gr}}\left(Y_0 \oplus_{\text{gr}} \int_{z_0}^{z} F\left(s,Y(s)\right)\Delta s, \hat{0}\right) \\
&\le \mathfrak{D}_{\text{gr}}\left(Y_0,\hat{0}\right) + \int_{z_0}^{z} \mathfrak{D}_{\text{gr}}\left(F\left(s,Y(s)\right),\hat{0}\right)\Delta s \\
&\le \mathfrak{D}_{\text{gr}}(Y_0,\hat{0}) + \int_{z_0}^{z} L\left(1+\mathfrak{D}_{\text{gr}}(Y(s),\hat{0})\right)\Delta s,
\end{aligned} \tag{4.13}$$

which yields that $\sup_{z\in[z_0,z_0+\lambda]_{\mathbb{T}}} \frac{1}{e_p(z,z_0)}\mathfrak{D}_{\text{gr}}\left(\mathcal{S}[Y](z),\hat{0}\right) \le \rho := \mathfrak{D}_{\text{gr}}(Y_0,\hat{0}) + L(1+\varrho)\lambda$. *Step 3*: The operator $\mathcal{S}$ maps bounded sets into equicontinuous sets in $C([z_0,z_0+\lambda]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}})$. One has

$$\mathfrak{D}_{\text{gr}}\left(\mathcal{S}[Y](z_1) \ominus_{\text{gr}} \mathcal{S}[Y](z_2),\hat{0}\right) = \mathfrak{D}_{\text{gr}}\left(\int_{z_0}^{z_1} F\left(s,Y(s)\right)\Delta s \ominus_{\text{gr}} \int_{z_0}^{z_2} F\left(s,Y(s)\right)\Delta s, \hat{0}\right)$$

$$= \mathfrak{D}_{\mathrm{gr}}\left(\int_{z_1}^{z_2} F(s,Y(s))\,\Delta s, \hat{0}\right) \le \int_{z_1}^{z_2} \mathfrak{D}_{\mathrm{gr}}\left(F(s,Y(s)),\hat{0}\right)\Delta s$$

$$\le \int_{z_1}^{z_2} L\left(1+\mathfrak{D}_{\mathrm{gr}}(Y(s),\hat{0})\right)\Delta s$$

$$\le L(1+\varrho)|z_1-z_2| \to 0 \tag{4.14}$$

whereas $z_1 \to z_2$. Thus, $\mathcal{S}$ is equicontinuous on $C([z_0,z_0+\lambda]_{\mathbb{T}},\mathscr{F}_{\mathbb{R}})$. Combining Steps 1, 2, and 3 with Arzelà–Ascoli theorem, we conclude that the map $\mathcal{S} : C([z_0,z_0+\lambda]_{\mathbb{T}},\mathscr{F}_{\mathbb{R}}) \to C([z_0,z_0+\lambda]_{\mathbb{T}},\mathscr{F}_{\mathbb{R}})$ is complete continuous. *Step 4*: A priori bounds. It requires us to show that the set $T = \{Y \in C([z_0,z_0+\lambda]_{\mathbb{T}},\mathscr{F}_{\mathbb{R}}) \mid Y = \beta\mathcal{S}Y,\ \beta\in[0,1)\}$ is bounded. Indeed, for $Y \in T$, we get

$$\sup_{z\in[z_0,z_0+\lambda]_{\mathbb{T}}} \mathfrak{D}_{\mathrm{gr}}\left(Y(z),\hat{0}\right) = \sup_{z\in[z_0,z_0+\lambda]_{\mathbb{T}}} \mathfrak{D}_{\mathrm{gr}}(\beta\mathcal{S}[Y](z),\hat{0})$$

$$= \sup_{z\in[z_0,z_0+\lambda]_{\mathbb{T}}} \frac{\mathfrak{D}_{\mathrm{gr}}\left(Y_0 \oplus_{\mathrm{gr}} \int_{z_0}^{z} F\left(s\,Y(s)\right)\Delta s,\hat{0}\right)}{e_L(z,z_0)}$$

$$\le \sup_{z\in[z_0,z_0+\lambda]_{\mathbb{T}}} \frac{\mathfrak{D}_{\mathrm{gr}}(Y_0,\hat{0})}{e_L(z,z_0)} + \sup_{z\in[z_0,z_0+\lambda]_{\mathbb{T}}} \frac{\int_{z_0}^{z} L\left(1+\mathfrak{D}_{\mathrm{gr}}(Y(s),\hat{0})\right)\Delta s}{e_L(z,z_0)}$$

$$\le \mathfrak{D}_{\mathrm{gr}}(Y_0,\hat{0}) + \sup_{z\in[z_0,z_0+\lambda]_{\mathbb{T}}} \frac{L(z-z_0) + \int_{z_0}^{z} Le_L(s,z_0)\frac{\mathfrak{D}_{\mathrm{gr}}(Y(s),\hat{0})}{e_L(s,z_0)}\Delta s}{e_L(z,z_0)}$$

$$\le \mathfrak{D}_{\mathrm{gr}}(Y_0,\hat{0}) + \sup_{z\in[z_0,z_0+\lambda]_{\mathbb{T}}} \frac{L(z-z_0) + \left\|Y\right\|_{\mathrm{gr}}^{L} \int_{z_0}^{z} Le_L(s,z_0)\Delta s}{e_L(z,z_0)}$$

$$\le \mathfrak{D}_{\mathrm{gr}}(Y_0,\hat{0}) + \sup_{z\in[z_0,z_0+\lambda]_{\mathbb{T}}} \frac{\left\|Y\right\|_{\mathrm{gr}}^{L}\left(e_L(z,z_0)-1\right) + L(z-z_0)}{e_L(z,z_0)}$$

$$= \le \mathfrak{D}_{\mathrm{gr}}(Y_0,\hat{0}) + L\lambda + \left\|Y\right\|_{\mathrm{gr}}^{L}\left(1-\frac{1}{e_L(z_0+\lambda,z_0)}\right). \tag{4.15}$$

Therefore, $\left\|Y\right\|_{\mathrm{gr}}^{L} \le e_L(z_0+\lambda,z_0)[L\lambda + \mathfrak{D}_{\mathrm{gr}}(Y_0,\hat{0})]$ which shows that Y is bounded. By Schaefer's theorem, we conclude that $\mathcal{S}$ has at least one fixed point. Hence problem (4.1) also has at least one solution, which completes the proof. □

Theorem 4.2. *Let $F : [z_0,z_0+\lambda]_{\mathbb{T}}^{\kappa} \times \mathscr{F}_{\mathbb{R}} \to \mathscr{F}_{\mathbb{R}}$ satisfy the conditions **(A)**–**(B)**. Then, there exists a unique solution to problem* (4.1)*. Moreover, assume that Y and Z are two any solutions of problem* (4.1) *corresponding to the input data $Y(z_0) = Y_0 \in \mathscr{F}_{\mathbb{R}}$ and $Z(z_0) = Z_0 \in \mathscr{F}_{\mathbb{R}}$, then the estimation*

$$\mathfrak{D}_{\mathrm{gr}}(Y(z),Z(z)) \le e_L(z,z_0)\mathfrak{D}_{\mathrm{gr}}\left(Y(z_0),Z(z_0)\right) \tag{4.16}$$

holds for all $z \in [z_0,z_0+\lambda]_{\mathbb{T}}$.

Proof. Set $p = L\theta$, where L is as in hypothesis (**B**) and $\theta > 1$. We consider the space $C([z_0,z_0+\lambda]_{\mathbb{T}},\mathscr{F}_{\mathbb{R}})$ endowed with the metric $\mathfrak{D}_{\mathrm{gr}}^{p}$ and the operator

$$\mathcal{S} : C([z_0,z_0+\lambda]_{\mathbb{T}},\mathscr{F}_{\mathbb{R}}) \to C([z_0,z_0+\lambda]_{\mathbb{T}},\mathscr{F}_{\mathbb{R}}) \tag{4.17}$$

$$Y \mapsto \mathcal{S}[Y](z) = Y_0 \oplus_{\mathrm{gr}} \int_{z_0}^{z} F(s,Y(s))\,\Delta s \ \text{ for all } z\in[z_0,z_0+\lambda]_{\mathbb{T}}. \tag{4.18}$$

Observe that, by Lemma 4.1, the solution of (4.1) is a fixed point of the operator $\mathcal{S}$. By the Banach fixed point theorem, we prove the existence and uniqueness of a fixed point of the operator $\mathcal{S}$. To this end, one shall verify that $\mathcal{S}$ is a contraction map. For arbitrary $Y,Z \in C([z_0,z_0+\lambda]_{\mathbb{T}},\mathscr{F}_{\mathbb{R}})$, one has

$$\mathfrak{D}_{\mathrm{gr}}(\mathcal{S}[Y],\mathcal{S}[Z]) = \mathfrak{D}_{\mathrm{gr}}\left(\int_{z_0}^{z} F(s,Y(s))\,\Delta s, \int_{z_0}^{z} F(s,Z(s))\,\Delta s\right)$$

$$\leq \int_{z_0}^{z} \mathfrak{D}_{\text{gr}}\left(F\left(s, Y(s)\right), F\left(s, Z(s)\right)\right) \Delta s$$

$$\leq \int_{z_0}^{z} L \mathfrak{D}_{\text{gr}}\left(Y(s), Z(s)\right) \Delta s, \tag{4.19}$$

where the first inequality follows from Theorem 3.5-ii) and the second from condition (**B**). For $p > 0$, we have $e_p(z, z_0) > 0$ for all $z \in [z_0, z_0 + \lambda]_{\mathbb{T}}$. Combining (4.19) with the property of supremum, it yields that

$$\begin{aligned}\sup_{z\in[z_0,z_0+\lambda]_{\mathbb{T}}} \frac{1}{e_p(z,z_0)} \mathfrak{D}_{\text{gr}}\left(\mathcal{S}[Y], \mathcal{S}[Z]\right) &\leq \sup_{z\in[z_0,z_0+\lambda]_{\mathbb{T}}} \left(\frac{1}{e_p(z,z_0)} \int_{z_0}^{z} L\mathfrak{D}_{\text{gr}}\left(Y(s), Z(s)\right) \Delta s \right) \\ &= L \sup_{z\in[z_0,z_0+\lambda]_{\mathbb{T}}} \left(\frac{1}{e_p(z,z_0)} \int_{z_0}^{z} e_p(s,z_0) \frac{\mathfrak{D}_{\text{gr}}\left(Y(s), Z(s)\right)}{e_p(s,z_0)} \Delta s \right) \\ &\leq L\mathfrak{D}^p_{\text{gr}}\left(Y, Z\right) \sup_{z\in[z_0,z_0+\lambda]_{\mathbb{T}}} \left(\frac{1}{e_p(z,z_0)} \int_{z_0}^{z} e_p(s,z_0) \Delta s \right). \end{aligned} \tag{4.20}$$

We observe that

$$\begin{aligned}\sup_{z\in[z_0,z_0+\lambda]_{\mathbb{T}}} \left(\frac{1}{e_p(z,z_0)} \int_{z_0}^{z} e_p(s,z_0) \Delta s \right) &= \sup_{z\in[z_0,z_0+\lambda]_{\mathbb{T}}} \left(\frac{1}{p e_p(z,z_0)} \int_{z_0}^{z} e_p^{\Delta}(s,z_0) \Delta s \right) \\ &= \sup_{z\in[z_0,z_0+\lambda]_{\mathbb{T}}} \frac{1}{p} \left(1 - \frac{1}{e_p(z,z_0)} \right) = \frac{1}{p} \left(1 - \frac{1}{e_p(z_0+\lambda,z_0)} \right). \end{aligned} \tag{4.21}$$

From the above, we receive

$$\begin{aligned}\mathfrak{D}^p_{\text{gr}}\left(\mathcal{S}[Y], \mathcal{S}[Z]\right) &\leq \frac{L}{p} \left(1 - \frac{1}{e_p(z_0+\lambda,z_0)} \right) \mathfrak{D}^p_{\text{gr}}\left(Y, Z\right) \\ &< \frac{1}{\theta} \mathfrak{D}^p_{\text{gr}}\left(Y, Z\right). \end{aligned} \tag{4.22}$$

Since $\theta > 1$, $\mathcal{S}$ is a contractive mapping in the space $(C([z_0, z_0 + \lambda]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}}), \mathfrak{D}^p_{\text{gr}})$. Therefore, by the Banach contraction principle, there is a unique function $Y \in C([z_0, z_0 + \lambda]_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}})$ such that $\mathcal{S}[Y] = Y$. Then, it follows that problem (4.1) admits a unique solution.

Now we assume that Y and Z are the solutions to the initial value problem (4.1) corresponding to the initial conditions Y_0 and Z_0. Denoting $\mathcal{W}(z) = \mathfrak{D}_{\text{gr}}\left(Y(z), Z(z)\right), z \in [z_0, z_0 + \lambda]_{\mathbb{T}}$, from Lemma 4.1 and hypothesis (**B**), one gets

$$\begin{aligned}\mathfrak{D}_{\text{gr}}\left(Y(z), Z(z)\right) &\leq \mathfrak{D}_{\text{gr}}\left(Y_0, Z_0\right) + \int_{z_0}^{z} \mathfrak{D}_{\text{gr}}\left(F\left(s, Y(s)\right), F\left(s, Z(s)\right)\right) \Delta s \\ &\leq \mathfrak{D}_{\text{gr}}\left(Y_0, Z_0\right) + L \int_{z_0}^{z} \mathfrak{D}_{\text{gr}}\left(Y(s), Z(s)\right) \Delta s \\ &= \mathfrak{D}_{\text{gr}}\left(Y_0, Z_0\right) + L \int_{z_0}^{z} \mathcal{W}(s) \Delta s. \end{aligned} \tag{4.23}$$

By applying Gronwall's inequality (see [25, Lemma 2.1.3]), one gets

$$\mathcal{W}(z) \leq e_L(z, z_0) \mathfrak{D}_{\text{gr}}\left(Y_0, Z_0\right) \quad \text{for all } z \in [z_0, z_0 + \lambda]_{\mathbb{T}}, \tag{4.24}$$

which completes the proof. □

5. Applications

This section examines the limitations of previous approaches in studying fuzzy dynamic equations on time scales and their application to modeling real-world phenomena. Additionally, we highlight the advantages of the method proposed in this paper for addressing these shortcomings. Comparative examples are also provided to illustrate the differences between the proposed approach and earlier methods. For convenience, we will recall some of the notations that were introduced in [20,23,42,43] as follows: ⊕ and

$\odot$ stand for the addition and the multiplication, respectively, in standard fuzzy arithmetic, and $X^{\Delta^{\text{gH}}}(z)$ means the delta generalized Hukuhara derivative of X at z.

5.1. The drawbacks of some previous approaches

Similar to research on fuzzy differential equations in continuous-time domains, initial value problems of fuzzy dynamic equations using the gH-difference approach, as well as the fuzzy standard interval arithmetic approach, also encounter several drawbacks. These limitations have been extensively discussed in the works of M. Mazandarani et al. [26–28] and N.V. Hoa et al. [18]. We will provide a brief summary of these shortcomings in the following categories.

1. *About the existence of the gH-difference.* The initial value problems of fuzzy dynamic equations on time scales using the gH-difference approach are generally not well-defined because the existence of the gH-difference is not guaranteed. This issue underscores the importance of careful analysis when working with fuzzy dynamic equations on time scales that involve the delta gH-derivative.
2. *About the existence of multi-solution.* When solving fuzzy dynamic equations on time scales involving the gH-difference approach, an initial assumption regarding the type of delta gH-differentiability (or the monotonicity of the solution) is made. It is crucial to recognize that depending on the specific choice of delta gH-differentiability, *multiple solutions* may arise for the same problem. This emphasizes the importance of careful analysis and interpretation of the obtained solutions.
3. *About doubling property*. Typically, solving fuzzy dynamic equations on time scales using the gH-difference approach requires addressing a system of real-valued dynamic equations for each type of delta gH-differentiability. This drawback, known as the *doubling property*, poses a significant challenge, particularly for fuzzy problems with dimensions greater than 2, as it necessitates solving a large number of real-valued dynamic equations to determine the solution to a given fuzzy problem.
4. *About factorization disability*. It is well known that property (iii) in Remark 2.2 does not hold, meaning $(\omega_1 \oplus \omega_2) \odot \omega_3 \neq \omega_1 \odot \omega_3 \oplus \omega_2 \odot \omega_3$, where $\omega_1, \omega_2, \omega_3 \in \mathscr{F}_{\mathbb{R}}$. This limitation, referred to as the *factorization disability*, introduces an unnatural aspect to fuzzy analysis calculations. More specifically, it is evident that the two equations below are not equivalent
$$Y^{\Delta^{\text{gH}}}(z) = \omega \odot (F(z) \oplus G(z)), \tag{5.1}$$
$$Y^{\Delta^{\text{gH}}}(z) = \omega \odot F(z) \oplus \omega \odot G(z), \tag{5.2}$$
for all $z \in [z_0, z_0 + \lambda]^{\kappa}_{\mathbb{T}}$, where $Y(z_0) = Y_0 \in \mathscr{F}_{\mathbb{R}}$, and F, G are fuzzy functions.
5. *About the unnatural behavior in modeling phenomenon (UBM)*. It is well known that $\omega_1 - \omega_2 = \omega_1 \oplus (-1) \odot \omega_2 \neq \hat{0}$ and $\omega_1 \oplus \omega_2 \neq \hat{0}$, for all $\omega_1, \omega_2 \in \mathscr{F}_{\mathbb{R}}$. Hence, we observe that there is no fuzzy solution to the initial value problems of fuzzy dynamic equations on time scales using the gH-difference approach
$$\begin{cases} Y^{\Delta^{\text{gH}}}(z) \oplus F(z, Y(z)) = \hat{0}, \quad \text{for all } z \in [z_0, z_0 + \lambda]^{\kappa}_{\mathbb{T}}, \\ Y(z_0) = Y_0 \in \mathscr{F}_{\mathbb{R}}. \end{cases} \tag{5.3}$$
This limitation is known as UBM and can significantly hinder the accurate representation of real-world phenomena.
6. *About the catastrophe of physics laws violation (CPLV)*. Given the drawbacks outlined in points 1 to 5, it is not feasible to accurately represent physical phenomena using fuzzy dynamic equations or fuzzy differential equations, nor to study the behavior of fuzzy dynamic equations through approaches based on the gH-difference and fuzzy standard interval arithmetic. For example, to analyze or predict the behavior of a physical phenomenon or dynamic equations that describe the above phenomenon, a unique solution is generally essential for informed decision-making. Practically, we require a unique fuzzy solution that does not depend on any specific assumptions about delta gH-differentiability or the monotonicity of the diameter. However, multiple solutions may arise when utilizing the gH-difference as well as fuzzy standard interval arithmetic. To deeply explain the concept of CPLV, we will quickly consider a widely recognized physical model, called Newton's Law of Cooling, to demonstrate that the above approach cannot be applied.
Newton's Law of Cooling explains how an object cools in an environment with a constant temperature. The rate of temperature change for the object is directly proportional to the difference between its temperature and the surrounding ambient temperature. Let $T(z)$ represent the temperature of the object at time z, and T_{env} denote the constant temperature of the surrounding environment. We will consider a real-valued dynamic equation on a time scale to describe Newton's Law of Cooling as follows:
$$T'(z) + \Lambda(T(z) - T_{\text{env}}) = 0, \quad \text{for all } z \in [z_0, z_0 + \lambda]^{\kappa}_{\mathbb{T}}, \tag{5.4}$$
with initial data $T(z_0) = T_0 \in \mathbb{R}$, where $T'(z)$ represents the rate of change of the temperature of the object over time; Λ is a positive constant that depends on the properties of the object and the environment; $-\Lambda(T(z) - T_{\text{env}})$ signifies that the rate of cooling is proportional to the difference between the object's temperature and the ambient temperature. The negative sign indicates that the object's temperature decreases over time as it cools. Observe that Equation (5.4) can be represented in the following equivalent forms: (i) $T'(z) = -\Lambda(T(z) - T_{\text{env}})$; (ii) $T'(z) + \Lambda T(z) = \Lambda T_{\text{env}}$; (iii) $T'(z) - \Lambda T_{\text{env}} = -\Lambda T(z)$. Furthermore, Equation (5.4) will have only one solution, and this solution also coincides with the solutions of forms (i)-(iii).
In contrast, when considering problem (5.4) within a fuzzy environment under the concept of gH-difference, the fuzzy standard interval arithmetic approach, and any related concepts to account for uncertainties in input data, such as $T_{\text{env}} \in \mathscr{F}_{\mathbb{R}}$ and $T_0 \in \mathscr{F}_{\mathbb{R}}$,

the uniqueness of the solution is no longer guaranteed. Additionally, the solutions to the forms (i)-(iii) and (5.4) do not coincide. This is a result of CPLV. To clarify the CPLV with more detailed analysis, we refer to the paper [26] in which an electrical circuit in a fuzzy environment is examined. The authors highlighted two key issues with CPLV: (1) It results in a violation of physical laws, which limits the applicability of fuzzy standard interval arithmetic methods and related concepts, such as the strongly generalized Hukuhara derivative, generalized Hukuhara derivative, and generalized derivative; (2) This violation of physical laws leads to the generation of infinite solutions by these approaches, as they fail to observe natural constraints.

Several promising approaches have recently emerged to address the aforementioned shortcomings. Two particularly effective methods stand out: one involves fuzzy arithmetic operations using the concept of linearly correlated fuzzy sets, proposed by E. Esmi et al. [14,36], and the other is based on granular operations, as proposed by A. Piegat et al. [33] and M. Mazandarani et al. [27]. However, despite its effectiveness and reliability compared to the gH-difference method, the linearly correlated difference approach still faces challenges. These challenges include the complexity of solving fuzzy dynamic equations and determining the joint possibility or correlation between fuzzy parameters, making it difficult to apply in the analysis of fuzzy dynamic equations. On the other hand, the approach based on the gr-difference, along with the granular operations presented in this paper, has proven effective in overcoming the limitations of previous methods. Inspired by granular operations, many recent achievements have been made across various scientific fields. Notable advancements include the study of Lyapunov stability in fuzzy dynamical systems using the gr-derivative [2,3], optimality in fuzzy optimization problems [29], and the finite-time stability of Caputo fractional fuzzy differential equations [37]. These results highlight the significant advantages of this method for studying and modeling practical problems compared to earlier approaches.

5.2. Numerical examples

To illustrate the benefits of using the granular operations approach for fuzzy dynamic equations on time scales in analyzing practical models, this subsection will compare it with a previously established method that examines fuzzy dynamic equations using the delta gH-derivative. This earlier method has been introduced and explored in various studies, as referenced in [20,23,42,43] and related works.

Example 5.1 *(The radioactive decay problem).* Radioactive decay encompasses the release of ionizing radiation from unstable elements, resulting in their conversion into different elements. This process occurs randomly, rendering it unfeasible to anticipate which atoms will decay at any specific instance. However, the foundational equation for radioactive decay is built on specific assumptions: (i) Any radioactive atom can decay at any time; (ii) The probability of decay is consistent across all atoms in the substance; (iii) The probability of decay remains constant over time; (iv) Atoms decay independently of each other. Very recently, investigating the mathematical model of radioactive decay on time scales has become crucial for gaining a more comprehensive and accurate understanding of the process. The benefits extend to a wide range of fields, including science, industry, health, and environmental management, where the precise knowledge of radioactive decay dynamics is essential for research, safety, and decision-making.

We consider the nuclear decay equation on an arbitrary time scale $\mathbb{T}$ under the form

$$\begin{cases} X^{\Delta^{\mathrm{gr}}}(z) = -pX(z), \quad z \in [0,\infty)_{\mathbb{T}}, \\ X(0) = v, \end{cases} \tag{5.5}$$

where $X(z)$ represents the count of radioactive nuclei within a particular radioactive material at time $z \in \mathbb{T}$ and $p > 0$ such that $-p \in \mathcal{R}(\mathbb{T},\mathbb{R})$; The quantity $p^{-1} > 0$ denotes the average lifetime of each radioactive atom, which is consistent for every type of radioactive substance; $X(0)$ signifies the initial quantity of radioactive nuclei. In this scenario, when there is ambiguity or lack of precise information regarding the initial quantity $X(0)$ of radioactive nuclei within the material, model (5.5) must incorporate uncertainty. Let us suppose that, based on a measurement, we have determined a fuzzy interval $[v]^r = [X_0 - \varepsilon_1(1-r), X_0 + \varepsilon_2(1-r)]$, where $0 < \varepsilon_1 \le X_0$ and $\varepsilon_2 > 0$, which covers the unknown exact value of X_0. In the nuclear decay model, we incorporate the concept of a fuzzy mapping denoted as X, where $X(z)$ can be viewed as a representation of the interval of uncertainty in the time z.

Next, the model (5.5) will be solved. From Definitions 2.4 and 2.5, the corresponding granular nuclear decay equation of problem (5.5) is given by

$$\begin{cases} \mathcal{H}(X^{\Delta^{\mathrm{gr}}}(z)) = \mathcal{H}(-pX(z)), \quad z \in [0,\infty)_{\mathbb{T}}, \\ \mathcal{H}(X(0)) = \mathcal{H}(v). \end{cases} \tag{5.6}$$

Utilizing Theorem 3.3, problem (5.6) can be represented equivalently by

$$\begin{cases} \dfrac{\partial X^{\mathrm{gr}}(z,r,\alpha_X)}{\Delta z} = -pX^{\mathrm{gr}}(z,r,\alpha_X), \quad z \in [0,\infty)_{\mathbb{T}}, \\ X^{\mathrm{gr}}(0,r,\alpha_X) = v^{\mathrm{gr}}(r,\alpha_v) \end{cases} \tag{5.7}$$

for all $r, \alpha_X, \alpha_v \in [0,1]$, where $v^{\mathrm{gr}}(r,\alpha_v) = X_0 - \varepsilon_1(1-r) + (\varepsilon_2 + \varepsilon_1)(1-r)\alpha_v$. By applying the method of variation of constants, the exact solution of (5.7) is obtained as follows:

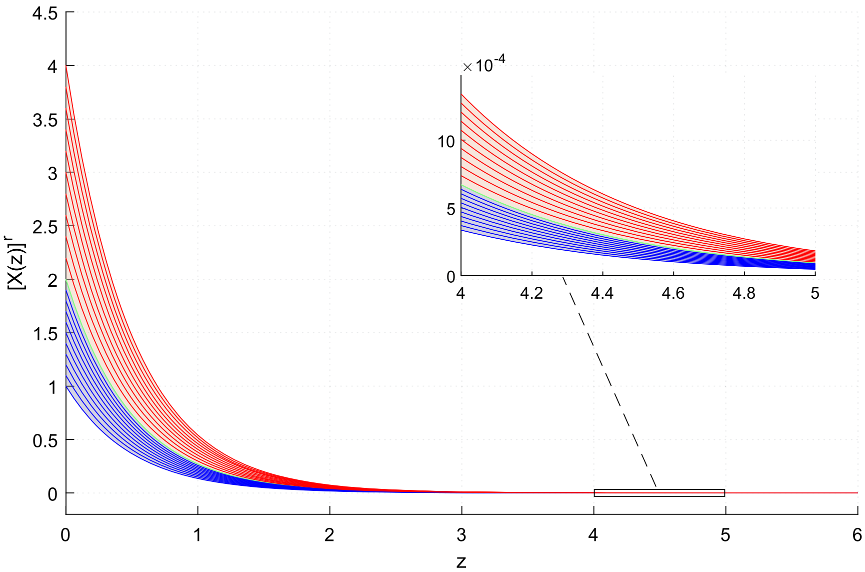

Fig. 9. The r-level set of $X(z)$ on $\mathbb{T}=\mathbb{R}$. The lower and upper borders of $[X(z)]^r$ are depicted by the blue-solid curves and red-solid curves, and the green-solid curve is for $r=1$.

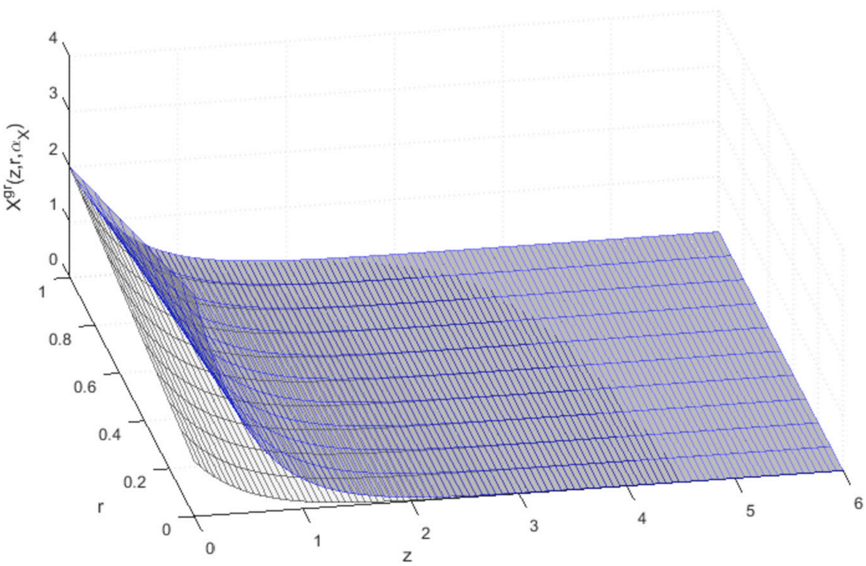

Fig. 10. The graph of $X^{\mathrm{gr}}(z,r,\alpha_X)$ on $\mathbb{T}=\mathbb{R}$, with $\alpha_X=0$ (the black grid) and $\alpha_X=1$ (the blue grid).

$$X^{\mathrm{gr}}(z,r,\alpha_X)=v^{\mathrm{gr}}(r,\alpha_v)e_{-p}(z,0). \tag{5.8}$$

Then, by the formula (2.6), we get

$$[X(z)]^r=\left[\inf_{\beta\geq r}\min_{\alpha_v\in[0,1]}\left(v^{\mathrm{gr}}(\beta,\alpha_v)e_{-p}(z,0)\right),\sup_{\beta\geq r}\max_{\alpha_v\in[0,1]}\left(v^{\mathrm{gr}}(\beta,\alpha_v)e_{-p}(z,0)\right)\right]. \tag{5.9}$$

Let us fix $X_0=2$, $\varepsilon_2=2\varepsilon_1=2$ and consider $\mathbb{T}=\mathbb{R}$, $p=2$. Then, the solution to problem (5.7) is given by

$$[X(z)]^r=[(1+r)e^{-2z},(4-2r)e^{-2z}]. \tag{5.10}$$

Trajectories of r-level set and granular representation of the solution to problem (5.7) are shown in Figs. 9 and 10. If we consider $\mathbb{T}=h\mathbb{Z}, h=0.2$, and $p=2$, then the solution to problem (5.7) is

$$[X(z)]^r=[(1+r)e_{-2}(z,0),(4-2r)e_{-2}(z,0)]. \tag{5.11}$$

Trajectories of the r-level set and granular representation of the solution to problem (5.7) are shown in Figs. 11 and 12.

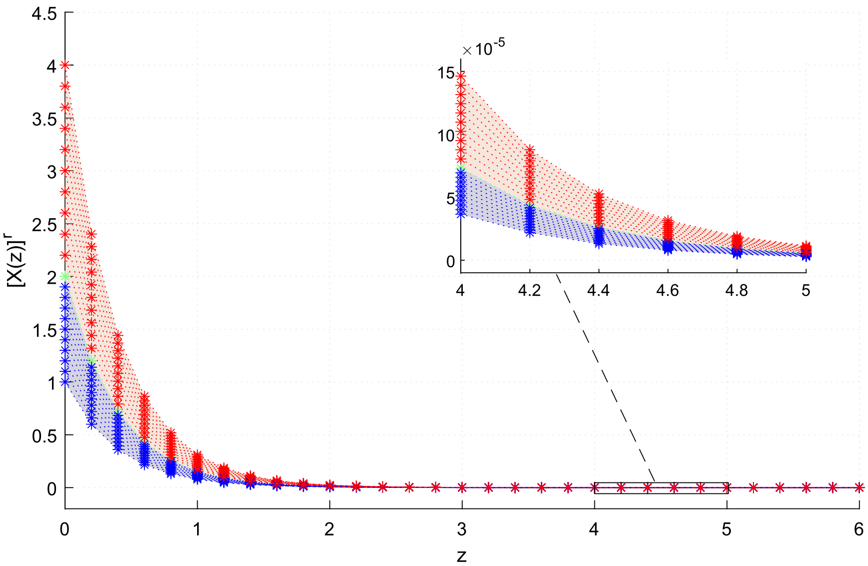

Fig. 11. The r-level set of $X(z)$ on $\mathbb{T} = 0.2\mathbb{Z}$. The lower and upper borders of $[X(z)]^r$ are shown as blue-dashed curves and red-dashed curves, and the green-dashed curve is with $r = 1$.

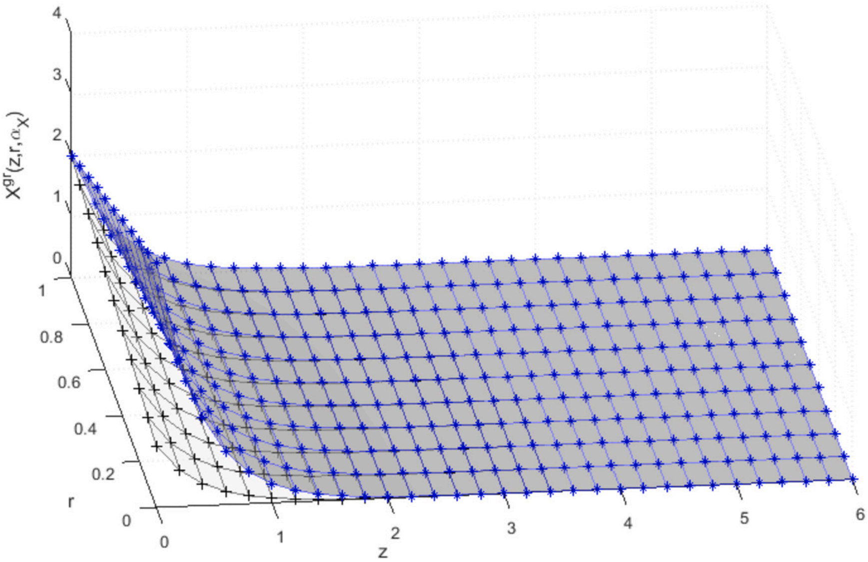

Fig. 12. The graph of $X^{\mathrm{gr}}(z, r, \alpha_X)$ on $\mathbb{T} = 0.2\mathbb{Z}$, with $\alpha_X = 0$ (the black-crossed grid) and $\alpha_X = 1$ (the blue-stared grid).

Remark 5.1. In our analysis, we compare the results obtained by applying the granular delta derivative in Example 5.1 with those derived using the recently introduced concept of the delta gH-derivative (Δ^{gH}-derivative), as presented in [20,23,42] and the references therein. Model (5.5) can be rewritten under the delta gH-derivative concept as

$$\begin{cases} X^{\Delta^{\mathrm{gH}}}(z) = -pX(z), \quad z \in [0, \infty)_{\mathbb{T}}, \\ X(0) = \upsilon, \end{cases} \tag{5.12}$$

where $p > 0$ such that $-p \in \mathcal{R}(\mathbb{T}, \mathbb{R})$; $X^{\Delta^{\mathrm{gH}}}(z)$ means the delta gH-derivative of X at z (see [23, Definition 9]). Using a method similar to the one mentioned in [23, Remark 10], with respect to the two types of Δ^{gH}-derivative of $X(z)$, one gets two solutions as

$$[X(z)]^r = \left[\frac{5-r}{2}e_{-p}(z,0) - \frac{3-3r}{2}e_p(z,0), \frac{5-r}{2}e_{-p}(z,0) + \frac{3-3r}{2}e_p(z,0)\right] \tag{5.13}$$

for all $z \in [0, \infty)_{\mathbb{T}}, r \in [0, 1]$, if the diameter of the r-level set of $X(z)$, $d([X(z)]^r := \overline{X}(z, r) - \underline{X}(z, r)$, is nondecreasing on $\mathbb{T}$, and

$$[X(z)]^r = \left[(1+r)e_{-p}(z,0), (4-2r)e_{-p}(z,0)\right] \tag{5.14}$$

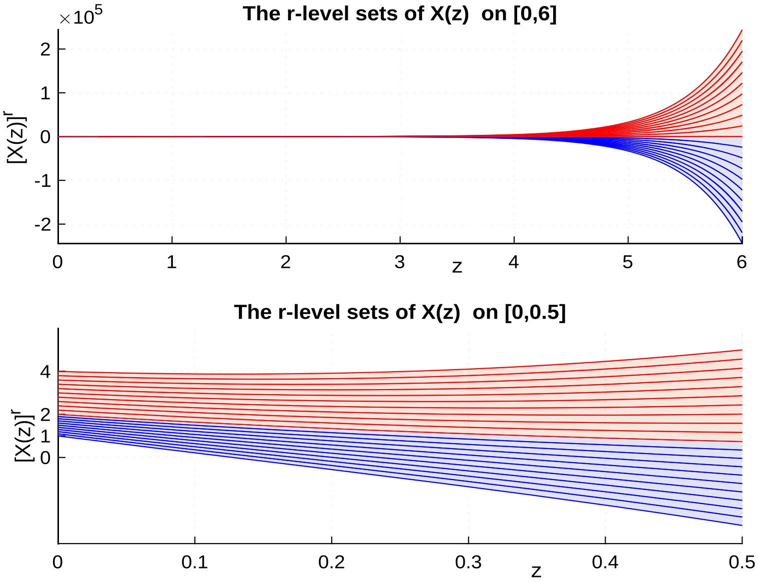

Fig. 13. The r-level set of $X(z)$ on $\mathbb{T} = \mathbb{R}$. The lower and upper borders of $[X(z)]^r$ are depicted by the blue-solid curves and red-solid curves.

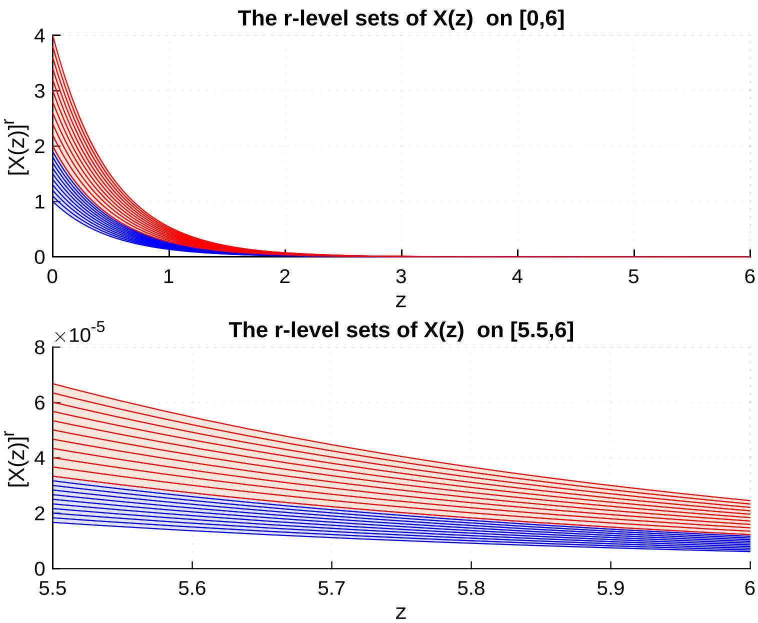

Fig. 14. The r-level set of $X(z)$ on $\mathbb{T} = \mathbb{R}$. The lower and upper borders of $[X(z)]^r$ are depicted by the blue-solid curves and red-solid curves.

if $d([X(z)]^r$ is nonincreasing on $\mathbb{T}$. The graphical representations of solutions to problem (5.12) with respect to $\mathbb{T} = \mathbb{R}$ and $p = 2$ are illustrated in Figs. 13 and 14. Through two cases of the solutions of the model (5.5), one gets the following typical observations:

- It is straightforward to verify that both (5.13) and (5.14) are valid solutions to the model (5.12), as $d([X(z)]^r) = (3 - 3r)e_p(z, 0)$ is nondecreasing on $[0, \infty)_{\mathbb{T}}$ in the form (5.13), and $d([X(z)]^r) = (3 - 3r)e_{-p}(z, 0)$ is nonincreasing on $[0, \infty)_{\mathbb{T}}$ in the form (5.14), respectively, where $p > 0$ and $r \in [0, 1]$. It is well known that since $X(z)$ represents the count of radioactive nuclei within a particular material at time z, it must be non-negative and non-increasing at any point. However, the r-level set of the solution to problem (5.12) given in (5.13) reveals that it is possible for $X(z)$ to take on negative values (see Fig. 13). In addition, there exist multiple solutions to the proposed model (5.12). These issues arise due to the CPLV limitations inherent in the gH-derivative approach.
- Consider Example 5.1, where the granular delta derivative approach offers significant flexibility by not requiring a specific type of differentiability or assumptions about the monotonicity of the solution's diameter during the solving process. This flexibility

effectively eliminates the issue of multiple solutions. Moreover, the drawbacks highlighted in Subsection 5.1 are also resolved. This feature provides a considerable advantage, making the approach more suitable for practical applications.

Example 5.2. Let us consider an initial value problem of fuzzy dynamics on an arbitrary time scale $\mathbb{T}$ under the form

$$\begin{cases} X^{\Delta^{\mathrm{gr}}}(z) = -pX(z) \oplus_{\mathrm{gr}} uz, \quad z \in [0,\infty)_{\mathbb{T}}, \\ X(0) = v, \end{cases} \tag{5.15}$$

where $u = (1, 1.5, 2), v = (0, 0.5, 1) \in \mathscr{F}_{\mathbb{R}}$ and $p > 0$ with $-p \in \mathcal{R}(\mathbb{T}, \mathbb{R})$. By direct computing, one gets $[u]^r = [1 + 0.5r, 2 - 0.5r]$ and $[v]^r = [0.5r, 1 - 0.5r]$. The corresponding HMFs of u and v are $u^{\mathrm{gr}}(r, \alpha_u) = 1 + 0.5r + (1 - r)\alpha_u$ and $v^{\mathrm{gr}}(r, \alpha_v) = 0.5r + (1 - r)\alpha_v$ for all $\alpha_u, \alpha_v \in [0, 1]$. Define $F(z, X(z)) := -pX(z) \oplus_{\mathrm{gr}} uz, z \in [0,\infty)_{\mathbb{T}}$. Then, one has

$$\mathfrak{D}_{\mathrm{gr}}\left(F(z, X(z)), F(z, Y(z))\right) = \mathfrak{D}_{\mathrm{gr}}\left(-pX(z) \oplus_{\mathrm{gr}} uz, -pY(z) \oplus_{\mathrm{gr}} uz\right) = |p|\mathfrak{D}_{\mathrm{gr}}\left(X(z), Y(z)\right) + |z|\mathfrak{D}_{\mathrm{gr}}(u, u). \tag{5.16}$$

Moreover, F belongs to $C_{\mathrm{rd}}([0,\infty)_{\mathbb{T}}, \mathscr{F}_{\mathbb{R}})$. Thus, the hypotheses of Theorem 4.2 hold, which yields that problem (5.15) has a unique solution. From Definitions 2.4 and 2.5, the corresponding granular dynamic equation of problem (5.15) is given by

$$\begin{cases} \mathcal{H}(X^{\Delta^{\mathrm{gr}}}(z)) = \mathcal{H}(-pX(z)) + \mathcal{H}(uz), \quad z \in [0,\infty)_{\mathbb{T}}, \\ \mathcal{H}(X(0)) = \mathcal{H}(v). \end{cases} \tag{5.17}$$

Using Theorem 3.3, the problem (5.17) can be equivalently rewritten as follows:

$$\begin{cases} \frac{\partial X^{\mathrm{gr}}(z,r,\alpha_X)}{\Delta z} = -pX^{\mathrm{gr}}(z, r, \alpha_X) + zu^{\mathrm{gr}}(r, \alpha_u), \quad z \in [0,\infty)_{\mathbb{T}}, \\ X^{\mathrm{gr}}(0, r, \alpha_X) = v^{\mathrm{gr}}(r, \alpha_v) \end{cases} \tag{5.18}$$

for all $r, \alpha_X, \alpha_u, \alpha_v \in [0, 1]$. The variation of constants method is used to obtain the solution to problem (5.18). For convenience, we put $q := -p < 0$. From the fact $X^{\mathrm{gr}}(z, r, \alpha_X) = X^{\mathrm{gr}}(\sigma(z), r, \alpha_X) - \mu(z)\frac{\partial X^{\mathrm{gr}}(z,r,\alpha_X)}{\Delta z}$, the first equation in (5.18) can be rewritten as

$$\frac{\partial X^{\mathrm{gr}}(z, r, \alpha_X)}{\Delta z} = qX^{\mathrm{gr}}(\sigma(z), r, \alpha_X) - q\mu(z)\frac{\partial X^{\mathrm{gr}}(z, r, \alpha_X)}{\Delta z} + zu^{\mathrm{gr}}(r, \alpha_u). \tag{5.19}$$

By dividing two sides of (5.19) by the term $1 + \mu(z)q$, one obtains

$$\frac{\partial X^{\mathrm{gr}}(z, r, \alpha_X)}{\Delta z} + (\ominus q)X^{\mathrm{gr}}(\sigma(z), r, \alpha_X) = \frac{zu^{\mathrm{gr}}(r, \alpha_u)}{1 + \mu(z)q}. \tag{5.20}$$

Multiplying both sides of (5.20) by $e_{\ominus q}(z, 0)$, one gets

$$\frac{\partial\left[X^{\mathrm{gr}}(z, r, \alpha_X)e_{\ominus q}(z, 0)\right]}{\Delta z} = \frac{zu^{\mathrm{gr}}(r, \alpha_u)}{1 + \mu(z)q}e_{\ominus q}(z, 0). \tag{5.21}$$

Next, we integrate equation (5.21) and derive

$$X^{\mathrm{gr}}(z, r, \alpha_X)e_{\ominus q}(z, 0) - X^{\mathrm{gr}}(0, r, \alpha_X)e_{\ominus q}(0, 0) = \int_0^z \frac{su^{\mathrm{gr}}(r, \alpha_u)}{1 + \mu(s)q}e_{\ominus q}(s, 0)\Delta s. \tag{5.22}$$

Using the equality $e_{\ominus q}(z, 0) = \frac{1}{e_q(z,0)}$ and multiplying both sides of (5.22) by $e_q(z, 0)$, we have

$$X^{\mathrm{gr}}(z, r, \alpha_X) = X^{\mathrm{gr}}(0, r, \alpha_X)e_q(z, 0) + u^{\mathrm{gr}}(r, \alpha_u)\int_0^z se_q(z, \sigma(s))\Delta s. \tag{5.23}$$

By direct computation, one gets $\int_0^z se_q(z, \sigma(s))\Delta s = \frac{1}{q^2}[e_q(z, 0) - 1] - \frac{1}{q}z$. Therefore, the solution to problem (5.18) is given by

$$X^{\mathrm{gr}}(z, r, \alpha_X) = v^{\mathrm{gr}}(r, \alpha_v)e_q(z, 0) + u^{\mathrm{gr}}(r, \alpha_u)\left(\frac{e_q(z, 0) - 1}{q^2} - \frac{1}{q}z\right). \tag{5.24}$$

Applying formula (2.6), we obtain

$$[X(z)]^r = \Bigg[\inf_{\beta\geq r}\min_{\alpha_u,\alpha_v}\left(v^{\mathrm{gr}}(\beta, \alpha_v)e_{-p}(z, 0) + u^{\mathrm{gr}}(\beta, \alpha_u)\left(\frac{e_{-p}(z, 0) - 1}{p^2} + \frac{1}{p}z\right)\right), \\ \sup_{\beta\geq r}\max_{\alpha_u,\alpha_v}\left(v^{\mathrm{gr}}(\beta, \alpha_v)e_{-p}(z, 0) + u^{\mathrm{gr}}(\beta, \alpha_u)\left(\frac{e_{-p}(z, 0) - 1}{p^2} + \frac{1}{p}z\right)\right)\Bigg]. \tag{5.25}$$

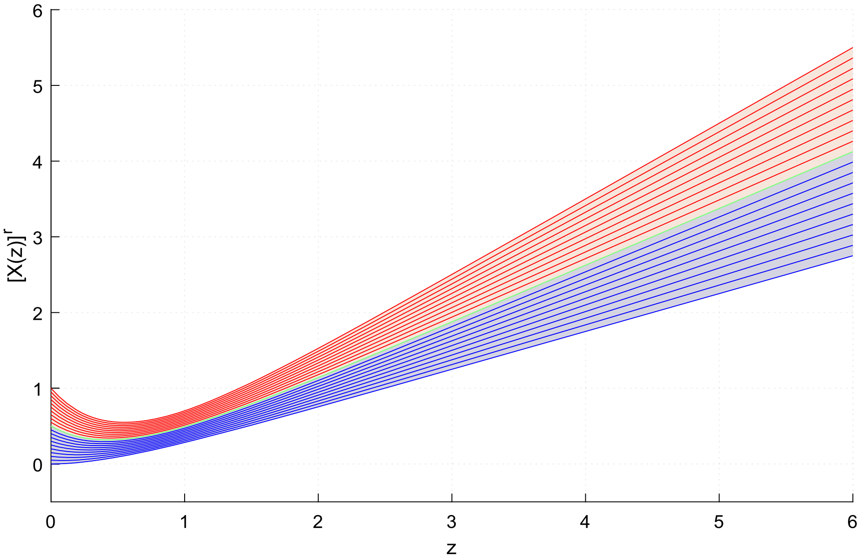

Fig. 15. The r-level set of $X(z)$ on $\mathbb{T}=\mathbb{R}$. The lower and upper borders of $[X(z)]^r$ are depicted by the blue-solid curves and red-solid curves, and the green-solid curve is for $r=1$.

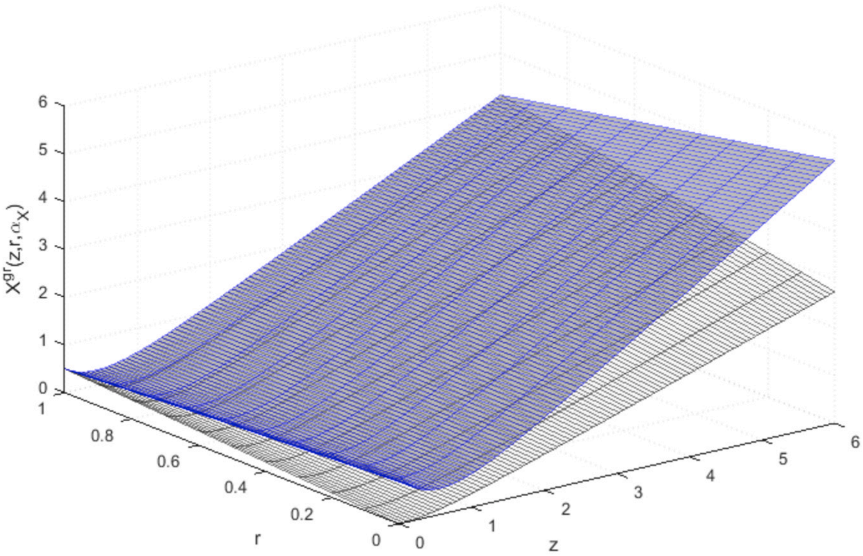

Fig. 16. The graph of $X^{\mathrm{gr}}(z,r,\alpha_X)$ on $\mathbb{T}=\mathbb{R}$, with $\alpha_X=0$ (the black grid) and $\alpha_X=1$ (the blue grid).

$$[X(z)]^r = [(0.25+0.625r)0.6^{5z}+(0.5z-0.25)(1+0.5r),(1.5-0.625r)0.6^{5z}+(0.5z-0.25)(2-0.5r)], \quad (5.26)$$

by utilizing the formula (2.6). Trajectories of the r-level set and granular representation of the solution to problem (5.15) are shown in Figs. 17 and 18. If $\mathbb{T}=\mathbb{R}$ and $p=2$, then the granular solution to problem (5.15) is provided by

$$X^{\mathrm{gr}}(z,r,\alpha_X)=[0.5r+(1-r)\alpha_v]e^{-2z}+[1+0.5r+(1-r)\alpha_u](0.25e^{-2z}+0.5z-0.25), \quad (5.27)$$

which leads to

$$[X(z)]^r=[(0.25+0.625r)e^{-2z}+(0.5z-0.25)(1+0.5r),(1.5-0.625r)e^{-2z}+(0.5z-0.25)(2-0.5r)], \quad (5.28)$$

by utilizing the formula (2.6). Trajectories of the r-level set and granular representation of the solution to problem (5.15) are shown in Figs. 15 and 16.

Similarly, if $\mathbb{T}=h\mathbb{Z}, h=0.2$, and $p=2$, then the granular solution to problem (5.15) is provided by

$$X^{\mathrm{gr}}(z,r,\alpha_X)=[0.5r+(1-r)\alpha_v]0.6^{5z}+[1+0.5r+(1-r)\alpha_u](0.25\cdot 0.6^{5z}+0.5z-0.25), \quad (5.29)$$

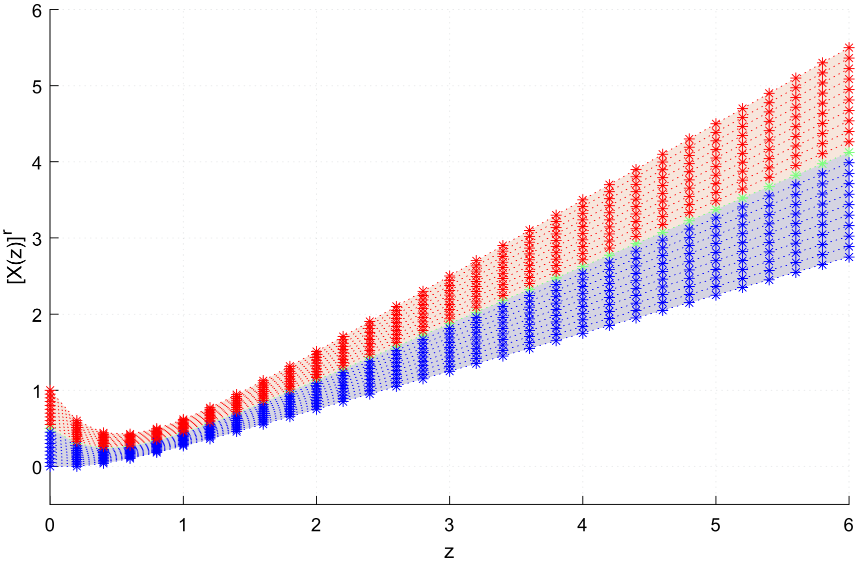

Fig. 17. The r-level set of $X(z)$ on $\mathbb{T} = 0.2\mathbb{Z}$. The lower and upper borders of $[X(z)]^r$ are shown as blue-dashed curves and red-dashed curves, and the green-dashed curve is for $r = 1$.

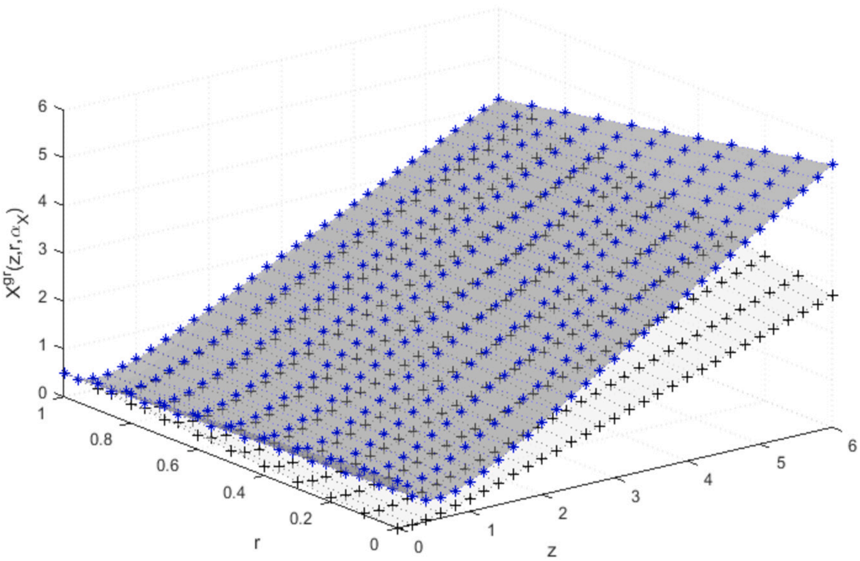

Fig. 18. The graph of $X^{\mathrm{gr}}(z, r, \alpha_X)$ on $\mathbb{T} = 0.2\mathbb{Z}$, with $\alpha_X = 0$ (the black-crossed grid) and $\alpha_X = 1$ (the blue-stared grid).

which leads to

Remark 5.2. Similar to Remark 5.1, we will also analyze Example 5.2 under the delta gH-derivative approach. Model (5.15) can be rewritten under the delta gH-derivative concept as

$$\begin{cases} X^{\Delta^{\mathrm{gH}}}(z) = -pX(z) \oplus uz, \quad z \in [0, \infty)_{\mathbb{T}}, \\ X(0) = v, \end{cases} \tag{5.30}$$

where $u = (1, 1.5, 2), v = (0, 0.5, 1) \in \mathscr{F}_{\mathbb{R}}$, $p > 0$ such that $-p \in \mathcal{R}(\mathbb{T}, \mathbb{R})$; $\oplus$ stands for the Minkowski addition, and $X^{\Delta^{\mathrm{gH}}}(z)$ means the delta generalized Hukuhara derivative of X at z (see [23, Definition 9]). Assume that $[X(z)]^r = [\underline{X}(z, r), \overline{X}(z, r)], z \in [0, \infty)_{\mathbb{T}}, r \in [0, 1]$. Using a method similar to the one mentioned in [23, Remark 10], with respect to the two types of Δ^{gH}-derivative of $X(z)$, we distinguish and solve the two problems

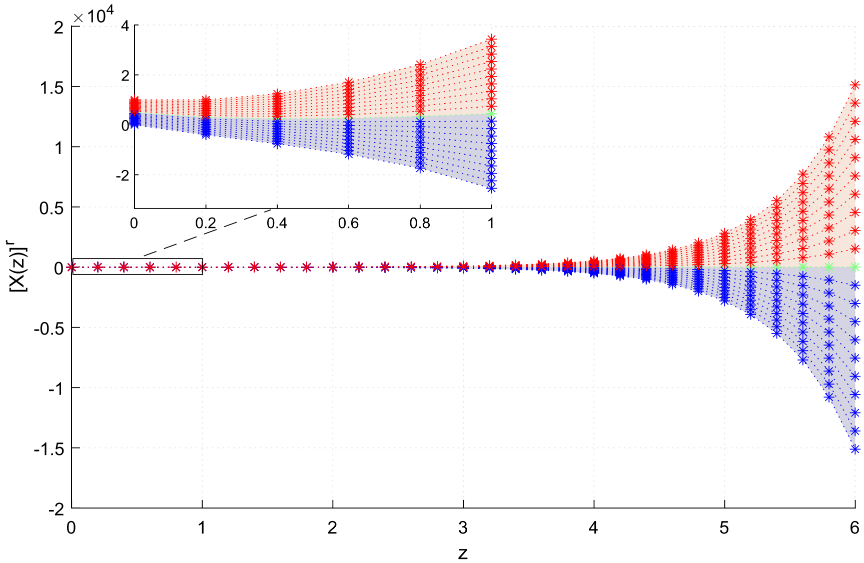

Fig. 19. The graphs of $\underline{X}(z,r)$ (the blue-dashed curves) and $\overline{X}(z,r)$ (the red-dashed curves) in different values of r (the green-dashed curve is with $r=1$).

$$\begin{cases} \underline{X}^{\Delta}(z,r) = -p\overline{X}(z,r) + z(1+0.5r), & \underline{X}(0,r) = 0.5r, \\ \overline{X}^{\Delta}(z,r) = -p\underline{X}(z,r) + z(2-0.5r), & \overline{X}(0,r) = 1-0.5r, \end{cases} \tag{5.31}$$

if the diameter of the r-level set of $X(z)$, $d([X(z)]^r := \overline{X}(z,r) - \underline{X}(z,r)$, is nondecreasing on $\mathbb{T}$, and

$$\begin{cases} \underline{X}^{\Delta}(z,r) = -p\underline{X}(z,r) + z(2-0.5r), & \underline{X}(0,r) = 0.5r, \\ \overline{X}^{\Delta}(z,r) = -p\overline{X}(z,r) + z(1+0.5r), & \overline{X}(0,r) = 1-0.5r, \end{cases} \tag{5.32}$$

if $d([X(z)]^r$ is nonincreasing on $\mathbb{T}$. We notice that, for $r=0$, problem (5.30) becomes $X^{\Delta^{\mathrm{gH}}}(z) = -pX(z) + z[1,2], z \in [0,\infty)_{\mathbb{T}}, X(0) = [0,1]$, what was rigorously studied in [23].

In order to find the solutions to system (5.31), we can transfer the system to a second-order dynamic equation as

$$\begin{cases} \underline{X}^{\Delta\Delta}(z,r) - p^2\underline{X}^{\Delta}(z,r) = -pz(2-0.5r) + 1 + 0.5r, \\ \underline{X}(0,r) = 0.5r, \quad \underline{X}^{\Delta}(0,r) = -p(1-0.5r). \end{cases} \tag{5.33}$$

Applying the method of variation of parameters in [9, Theorem 3.73], one receives the exact solution to (5.33) as

$$\underline{X}(z,r) = \frac{p^2+3}{2p^2} e_{-p}(z,0) + \frac{(p^2+1)(r-1)}{2p^2} e_p(z,0) + \frac{2-0.5r}{p} z - \frac{1+0.5r}{p^2} \tag{5.34}$$

for all $z \in [0,\infty)_{\mathbb{T}}, r \in [0,1]$. Similarly, we obtain

$$\overline{X}(z,r) = \frac{p^2+3}{2p^2} e_{-p}(z,0) + \frac{(p^2+1)(1-r)}{2p^2} e_p(z,0) + \frac{1+0.5r}{p} z - \frac{2-0.5r}{p^2}. \tag{5.35}$$

The trajectories of $\underline{X}(z,r)$, $\overline{X}(z,r), z \in [0,6]_{\mathbb{T}}$ in the case that $\mathbb{T} = hZ, h = 0.2$ and $p=2$ are shown in Fig. 19. Apparently, there is no switching point between $\underline{X}(z,r)$ and $\overline{X}(z,r), z \in [0,6]_{\mathbb{T}}$. Furthermore, the diameter of the r-level set of $X(z)$ is also increasing on $\mathbb{T}$. Therefore, one can deduce that $X(z)$ is a solution to problem (5.30), which has the form $[X(z)]^r = [\underline{X}(z,r), \overline{X}(z,r)]$ for all $z \in [0,6]_{\mathbb{T}}$, where $\underline{X}(z,r)$ and $\overline{X}(z,r)$ are given by (5.34) and (5.35), respectively.

Next, to find the solution to system (5.32), we employ the method of variation of parameters (see [9, Theorem 2.77]). The exact solution to (5.32) is given as follows:

$$\underline{X}(z,r) = \left(0.5r + \frac{2-0.5r}{p^2}\right) e_{-p}(z,0) + \frac{(2-0.5r)z}{p} - \frac{2-0.5r}{p^2} \tag{5.36}$$

and

$$\overline{X}(z,r) = \left(1 - 0.5r + \frac{1+0.5r}{p^2}\right) e_{-p}(z,0) + \frac{(1+0.5r)z}{p} - \frac{1+0.5r}{p^2} \tag{5.37}$$

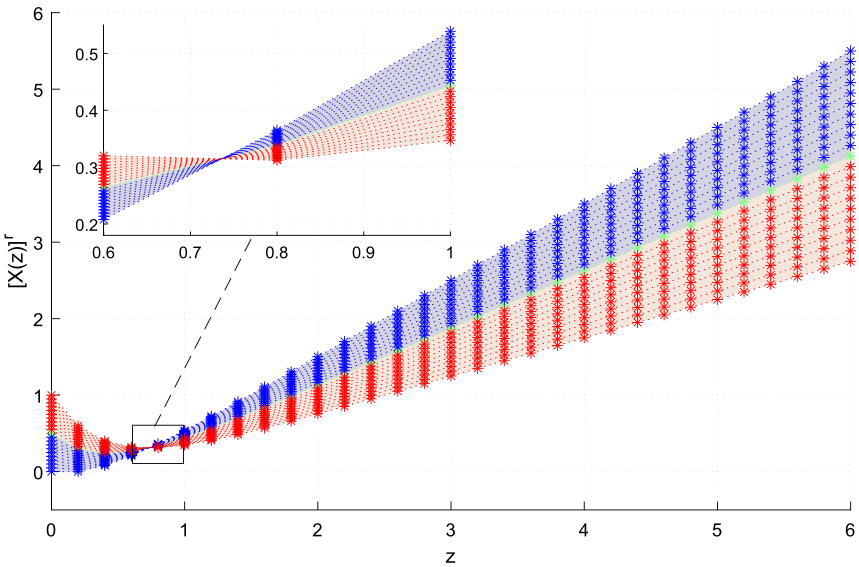

Fig. 20. The graphs of $\underline{X}(z,r)$ (the blue-dashed curves) and $\overline{X}(z,r)$ (the red-dashed curves) in different values of r (the green-dashed curve is with $r=1$).

for all $z \in [0,\infty)_{\mathbb{T}}$, $r \in [0,1]$. The trajectories of $\underline{X}(z,r)$ and $\overline{X}(z,r)$, $z \in [0,6]_{\mathbb{T}}$ in the case that $\mathbb{T} = h\mathbb{Z}$, $h = 0.2$ and $p = 2$ are illustrated in Fig. 20. Apparently, one observes that there is a switching point between $\underline{X}(z,r)$ and $\overline{X}(z,r)$, $z \in [0,2]_{\mathbb{T}}$ and it is not on $[0,6]_{\mathbb{T}}$. This shows that the diameter $d([X(z)]^r)$ of the r-level set of $X(z)$ is nondecreasing on $[0,6]_{\mathbb{T}}$, which implies that X is not a solution of problem (5.30).

Through two cases of the solutions of problem (5.30), one gets the following typical observations:

- Problem (5.30) has a solution with a nondecreasing diameter of the r-level set within the interval $[0,6]_{\mathbb{T}}$, as it is demonstrated in Fig. 19. However, this state of the system may not accurately represent the behavior of the system in practical situations. Typically, the solution to problem (5.30) in a crisp (nonfuzzy) context is always positive for all $z \in [0,6]_{\mathbb{T}}$, given positive input data. This condition is not satisfied by $X(z)$, as depicted in Fig. 19, where for sufficiently large values of z, the values of $X(z)$ can become nonpositive. This is due to the CPLV that the gH-derivative approach suffers from. On the other hand, when we employ the granular delta derivative concept, as demonstrated in Example 5.2, the fuzzy solution remains consistently positive and accurately mirrors the system's behavior, as shown in Figs. 15 and 17.
- When we utilize the granular delta derivative approach, the flexibility of not being restricted to a specific type of differentiability (or the assumption of monotonicity of the diameter of the solution) during the solving process eliminates the problem of multiple solutions. Rather than having to solve two systems (5.31) and (5.32), as seen in this example, we only need to solve a single system, as in Example 5.2. This characteristic is highly advantageous and is better suited for practical applications.

Example 5.3 *(Malthusian model on hybrid domains)*. Let us consider the special time scale

$$\mathbb{T} = \bigcup_{m=0}^{\infty} [2m, 2m+1]. \tag{5.38}$$

This time scale consists of an infinite number of disjoint closed intervals, making it suitable for modeling and studying real-world phenomena on both continuous and discrete time domains simultaneously (see [9]). In this example, the fuzzy Malthusian model on the hybrid domain (5.38) takes the form:

$$P^{\Delta^{\mathrm{gr}}}(z) = \varrho(z)P(z), \quad P(0) = P_0, \tag{5.39}$$

where P_0 is a fuzzy initial population at time $z = 0$, $\varrho : \mathbb{T} \to \mathbb{R}$ is the real-valued growth function defined by

$$\varrho(z) = \begin{cases} -k & \text{if } z \in [2m, 2m+1), \\ \lambda & \text{if } z = 2m+1, \end{cases} \tag{5.40}$$

with $m \geq 0$ and $\lambda, k \in (0,1)$.

Define $F : \mathbb{T} \times \mathscr{F}_{\mathbb{R}} \to \mathscr{F}_{\mathbb{R}}$ with $F(z, P(z)) := \varrho(z)P(z)$ and set $n_0 = \max_{z\in\mathbb{T}} |\varrho(z)|$. For problem (5.39), we examine the validity of conditions in Theorem 4.2. Indeed, because F is rd-continuous, hypothesis (**A**) holds. Moreover, for any $P, Q \in C_{\mathrm{rd}}(\mathbb{T}, \mathscr{F}_{\mathbb{R}})$, one has

$$\mathfrak{D}_{\mathrm{gr}}(F(z, P(z)), Q(z, P(z))) = \mathfrak{D}_{\mathrm{gr}}(\varrho(z)F(z), \varrho(z)Q(z)) \tag{5.41}$$

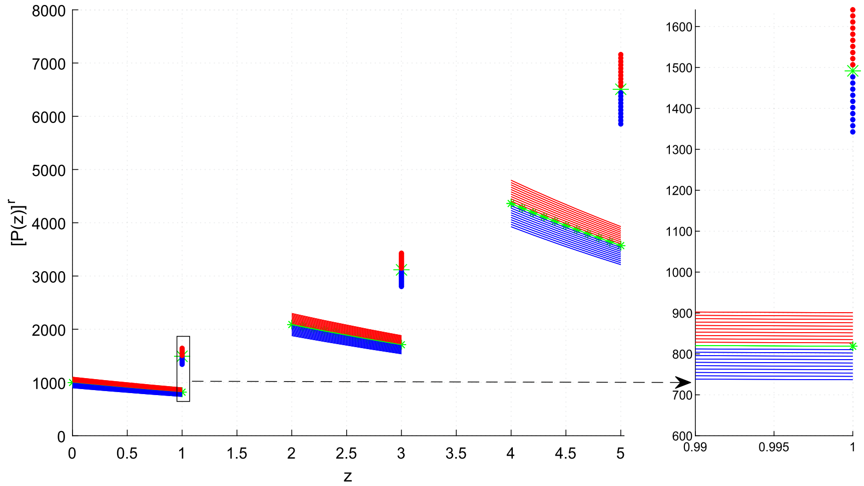

Fig. 21. The r-level set of $P(z)$ on $\mathbb{T}$ with the three first values of m. The lower and upper borders of $[P(z)]^r$ are depicted by the blue-solid curves and red-solid curves, and the green-stared curve is with $r = 1$.

$$= |\varrho(z)|\mathfrak{D}_{\rm gr}(F(z), Q(z)) \tag{5.42}$$

$$\leq n_0 \mathfrak{D}_{\rm gr}(F(z), Q(z)), \tag{5.43}$$

which means that (**B**) is fulfilled with $L = n_0$. From Definitions 2.4 and 2.5, the corresponding granular dynamic equation of (5.40) is represented by

$$\begin{cases} \frac{\partial P^{\rm gr}(z,r,\alpha_P)}{\Delta z} = \varrho(z) P^{\rm gr}(z,r,\alpha_P), \\ P^{\rm gr}(0,r,\alpha_P) = P_0^{\rm gr}(r,\alpha_{P_0}), \end{cases} \tag{5.44}$$

for all $r, \alpha_P, \alpha_{P_0} \in [0,1]$. Utilizing induction method, one gets the solution

$$P^{\rm gr}(z,r,\alpha_P) = \begin{cases} (1+\lambda)^m P_0^{\rm gr}(r,\alpha_{P_0}) e^{\lambda m - k(z-2m)} & \text{if } z \in [2m, 2m+1), \\ (1+\lambda)^m P_0^{\rm gr}(r,\alpha_{P_0}) e^{\lambda(z-m)} & \text{if } z = 2m+1. \end{cases} \tag{5.45}$$

The r-level set of the solution to problem (5.39) is obtained by formula (2.6). To illustrate problem (5.39), we choose $k = 0.2$, $\lambda = 0.4$, and the population size P_0 at time $z = 0$ to be $P_0 = (900, 1000, 1100) \in \mathscr{F}_{\mathbb{R}}$. In this case, the exact solution to problem (5.39) is given by

$$P^{\rm gr}(z,r,\alpha_P) = \begin{cases} 1.4^m[900 + 100r + (200 - 200r)\alpha_{P_0}]e^{0.4m - 0.2(z-2m)} & \text{if } z \in [2m, 2m+1), \\ 1.4^m[900 + 100r + (200 - 200r)\alpha_{P_0}]e^{0.4(z-m)} & \text{if } z = 2m+1 \end{cases} \tag{5.46}$$

and the r-level set of $P(z)$ is

$$[P(z)]^r = \begin{cases} \left[(900+100r)1.4^m e^{0.8m-0.2z}, (1100-100r)1.4^m e^{0.8m-0.2z}\right] & \text{if } z \in [2m, 2m+1), \\ \left[(900+100r)1.4^m e^{0.4(z-m)}, (1100-100r)1.4^m e^{0.4(z-m)}\right] & \text{if } z = 2m+1. \end{cases} \tag{5.47}$$

The r-level set and granular representation of the solution to problem (5.39) are shown in Figs. 21 and 22, respectively.

6. Conclusion

In this paper, we introduced the novel concept of granular delta differentiability and integrability for fuzzy functions on time scales, leveraging granular arithmetic operations within multidimensional fuzzy arithmetic (MFA). This concept allowed us to establish the fundamental principles of fuzzy calculus on time scales. By applying Schaefer's fixed-point theorem, we rigorously proved the existence of solutions to fuzzy dynamic equations. Furthermore, we explored the existence of a unique solution and the continuous dependence of the solution on the initial conditions of the fuzzy dynamic equations.

Stability analysis of dynamic systems on time scales with uncertainties is a crucial area of research with applications in various fields such as control theory, engineering, mathematical biology, etc. For further works, using the foundations of fuzzy calculus on time scales in this paper, we shall investigate the stability (in different types) properties of dynamic systems on time scales with uncertain information. For instance, we shall consider fuzzy switched impulsive dynamical systems on time scales as follows:

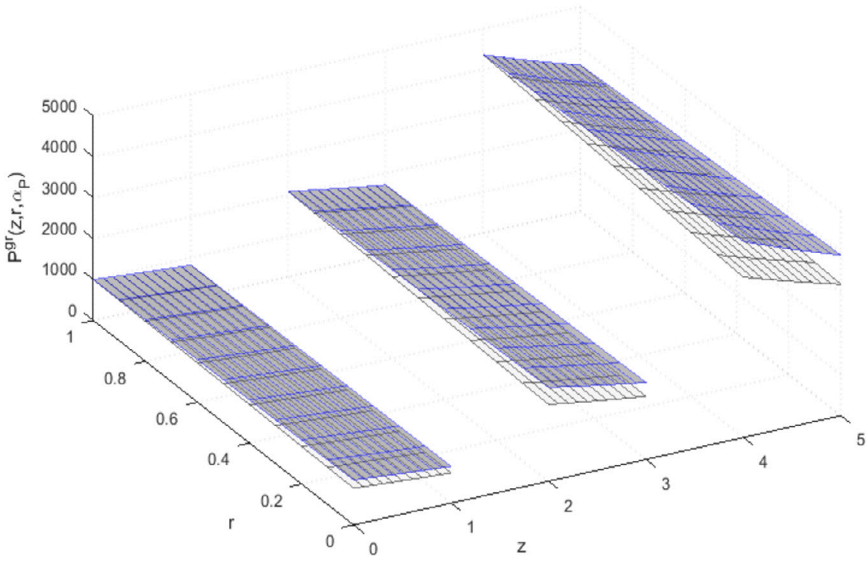

Fig. 22. The graph of $P^{\mathrm{gr}}(z, r, \alpha_P)$ with $\alpha_P = 0$ (the black grid) and $\alpha_P = 1$ (the blue grid).

$$\begin{cases} Y^{\Delta^{\mathrm{gr}}}(t) = A_k Y(t), & t \in (t_{k-1}, t_k]_{\mathbb{T}},\ k = 1, 2, \dots, \\ Y(t_k^+) = F_k(t_k, Y(t_k)), & k = 1, 2, \dots, \\ Y(0) = Y_0, \end{cases} \tag{6.1}$$

and impulsive system with a fuzzy control

$$\begin{cases} Y^{\Delta^{\mathrm{gr}}}(t) = A_k Y(t) + B_k \Psi(t), & t \in (t_{k-1}, t_k]_{\mathbb{T}},\ k = 1, 2, \dots, \\ Y(t_k^+) = F_k(t_k, Y(t_k)), & k = 1, 2, \dots, \\ Y(0) = Y_0, \end{cases} \tag{6.2}$$

where $\mathbb{T}$ is an arbitrary time scale, $Y \in \mathscr{F}_{\mathbb{R}}^n$ is a fuzzy state variable, and $A_k \in C_{\mathrm{prd}}(\mathbb{T}, \mathscr{F}_{\mathbb{R}}^n)$, $B_k \in C_{\mathrm{prd}}(\mathbb{T}, \mathscr{F}_{\mathbb{R}}^{n\times m})$, $\Psi \in \mathscr{F}_{\mathbb{R}}^m$ is a fuzzy control function, $Y(t_k^+)$ and $Y(t_k^-)$ stand for the right and the left limit of $Y(t)$ at $t = t_k$, and F_k are the continuous fuzzy functions that satisfy some further specific conditions. We will examine an impulsive switched system on time scales and obtain finite-time stability results for a such problem by constructing a common fuzzy Lyapunov quadratic function on time scales.

CRediT authorship contribution statement

Tri Truong: Writing – review & editing, Writing – original draft, Visualization. **Martin Bohner:** Writing – review & editing, Conceptualization. **Ewa Girejko:** Writing – review & editing, Conceptualization. **Agnieszka B. Malinowska:** Writing – review & editing, Supervision, Conceptualization. **Ngo Van Hoa:** Writing – review & editing, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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