INTERNATIONAL JOURNAL OF COMPUTER MATHEMATICS
https://doi.org/10.1080/00207160.2022.2090836

Taylor & Francis
Taylor & Francis Group

Check for updates

RESEARCH ARTICLE

Numerical study of generalized modified Caputo fractional differential equations

Imran Talib [a] and Martin Bohner[b]

[a]Nonlinear Analysis Group (NAG), Mathematics Department, Virtual University of Pakistan, Lahore, Pakistan;
[b]Missouri University of Science and Technology, Rolla, MO, USA

ABSTRACT

In the present study, we introduce two new operational matrices of fractional Legendre function vectors in the sense of generalized Caputo-type fractional derivative and generalized Riemann–Liouville-type fractional integral operators. The derivative and integral operational matrices developed in the sense of Caputo and Riemann–Liouville operators are special cases of our proposed generalized operational matrices for $\beta, \eta = 1$. Then, we present a numerical method that is dependent on the generalized derivative and integral operational matrices. The applicability and accuracy of the presented method is tested by solving various problems and then comparing the results obtained otherwise by using various numerical methods including spectral collocation methods, spectral Tau method, stochastic approach, and Taylor matrix approach. Moreover, our presented method transforms the problems into Sylvester equations that are easily solvable by using MATLAB or MATHEMATICA. We believe that the newly derived generalized operational matrices and the presented method are expected to be further used to formulate and simulate many generalized Caputo-type fractional models.

ARTICLE HISTORY
Received 5 April 2022
Revised 30 May 2022
Accepted 10 June 2022

KEYWORDS
Generalized operational matrices; generalized fractional derivative in Caputo sense; Sylvester equations; orthogonal polynomials; Legendre function vectors

2022 AMS SUBJECT CLASSIFICATIONS
65L05; 26A33; 35R11

1. Introduction

Historically on 30 September 1695, the intellectual debate between two renowned mathematicians, L'Hospital and Leibniz gave birth to the topic of fractional calculus. For many years, it was considered a mathematical curiosity without applications in physical sciences and other science-related disciplines. In 1823, the first notable application of fractional operators had been reported in Abel's solution to the so-called tautochrone problem: finding the curve such that the time needed for a particle to descend from a given position to the bottom of the curve (assuming there is no friction) is independent of position (see [21]). However, the topic which was predicted by Leibniz a paradox has nowadays evolved and attracted the interests of many researchers working in various disciplines of engineering and sciences (see [5–7,15,23]).

The fractional derivative has not a unique definition like classical derivatives. Various fractional derivative operators, such as Riemann–Liouville (RL), Hadamard, Caputo, Hilfer, and many others have been successfully utilized in solving various problems of mathematics. Among them, the most studied operators are RL and Caputo that include fractional integrals. The fractional integral operator

CONTACT Imran Talib imrantaalib@gmail.com; imrantalib@vu.edu.pk Nonlinear Analysis Group (NAG), Mathematics Department, Virtual University of Pakistan, Pakistan

in RL sense is defined as

$$ {}_{\mathrm{RL}}J^{\delta}_{a^+}u(z)=\frac{1}{\Gamma(\delta)}\int_a^z (z-x)^{\delta-1}u(x)\,\mathrm{d}x,\quad z>a,\ \delta>0. $$

Consequently, the fractional derivative operators in RL and Caputo sense are defined as

$$ \begin{aligned} {}_{\mathrm{RL}}\mathcal{D}^{\delta}_{a^+}u(z) &= D^n{}_{\mathrm{RL}}J^{n-\delta}_{a^+}u(z)\\ &=\frac{1}{\Gamma(n-\delta)}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\int_a^z (z-x)^{n-\delta-1}u(x)\,\mathrm{d}x,\quad z>a,\\ {}_{\mathrm{C}}\mathcal{D}^{\delta}_{a^+}u(z) &= {}_{\mathrm{RL}}J^{n-\delta}_{a^+}D^n u(z)\\ &=\frac{1}{\Gamma(n-\delta)}\int_a^z (z-x)^{n-\delta-1}u^{(n)}(x)\,\mathrm{d}x,\quad z>a, \end{aligned} \tag{1} $$

respectively, where $n-1<\delta<n$, $n\in\mathbb{N}$, and $\delta>0$. However, the fractional derivative operator in Caputo sense has been preferably used in the fractional modelling of physical phenomena due to its compatibility to model them with integer-order initial or boundary conditions. In addition to that, it has some properties similar to integer-order derivatives. For instance, the Caputo operator satisfies

$$ {}_{\mathrm{RL}}J^{\delta}_{a^+}{}_{\mathrm{C}}\mathcal{D}^{\delta}_{a^+}u(z)=u(z)-\sum_{l=0}^{n-1}\frac{u^{(l)}(a)}{l!}(z-a)^l,\quad z>a,\ n-1<\delta<n, \tag{2} $$

and

$$ {}_{\mathrm{C}}\mathcal{D}^{\delta}_{a^+}B=0,\ B \text{ is constant.} \tag{3} $$

Recently, generalized fractional integral operators (GFIO) were defined by introducing the fractional integral of a given function related to another function in the following way (see [17,20,26]).

Definition 1.1: The GFIO of order $\delta>0$ of the function u is defined as

$$ J^{\delta,\eta}_{a^+}u(z)=\frac{\eta^{1-\delta}}{\Gamma(\delta)}\int_a^z x^{\eta-1}(z^\eta-x^\eta)^{\delta-1}u(x)\,\mathrm{d}x,\quad z>a,\ \eta>0, \tag{4} $$

provided the integral exists.

Accordingly, the corresponding generalized fractional derivative operators (GFDO) in RL and Caputo type sense for $n-1<\delta<n$, $n\in\mathbb{N}$ can be defined as follows.

Definition 1.2: The GFDO of order $\delta>0$ of a function u in RL type sense is defined as (see [18])

$$ {}_{\mathrm{RL}}\mathcal{D}^{\delta,\eta}_{a^+}u(z)=\frac{\eta^{\delta-n+1}}{\Gamma(n-\delta)}\left(z^{1-\eta}\frac{\mathrm{d}}{\mathrm{d}z}\right)^n\int_a^z x^{\eta-1}(z^\eta-x^\eta)^{n-\delta-1}u(x)\,\mathrm{d}x,\quad z>a\geq 0,\ \eta>0. \tag{5} $$

Definition 1.3: The GFDO of order $\delta>0$ of a function u in Caputo-type sense is defined as (see [16])

$$ {}_{\mathrm{C}}\mathcal{D}^{\delta,\eta}_{a^+}u(z)=\left({}_{\mathrm{RL}}\mathcal{D}^{\delta,\eta}_{a^+}\left[u(t)-\sum_{l=0}^{n-1}\frac{u^{(l)}(a)}{l!}(t-a)^l\right]\right)(z),\quad z>a\geq 0,\ \eta>0. \tag{6} $$

On the other hand, to simulate fractional models numerically, spectral methods are efficient, reliable, and stable numerical tools that have been implemented to solve numerically various types of fractional differential equations (FDEs) that include Caputo derivative, see [4,10,14,25,27,28,30].

The framework of these methods is based on fractional-order derivative operational matrices or fractional-order integral operational matrices of orthogonal polynomials.

Generalization is a very common and vital phenomenon in mathematics. We are always interested in finding an abstract structure that can be analysed for itself and which covers many useful examples as special cases. So, in this study, we introduce generalized algorithms which cover the results discussed in [29] as special cases. Specifically, the fractional integration operational matrix [29, Theorem 4.5] and the fractional differentiation operational matrix [29, Theorem 4.7] are particular cases of our proposed operational matrices for $\eta = 1$. In addition, our proposed numerical algorithm is capable to solve fractional models which are based either on Caputo-fractional derivative or modified generalized Caputo-fractional derivative.

Based on the generalized proposed algorithms, our first motivation in this study is to solve the FDEs that include the modified GFDO of Caputo type introduced by Odibat and Baleanu in 2020 (see [22]). The second motivation is to introduce a generalized numerical method which covers the results of [29] as special cases and produces more accurate numerical results as compared to spectral methods, like spectral Tau and spectral collocation. The framework of our presented method is somewhat similar to spectral methods, however, the need of the residual functions and the collocation points are not required in our approach to generate the system of algebraic equations like spectral Tau and spectral collocation methods. So by implementing the presented approach, we can transform the problems into algebraic equations of Sylvester type without computing the residuals and collocating the equations at suitable collocation points. Consequently, we obtain better accuracy in the approximate results as compared to the other numerical techniques, like spectral Tau method [25], function approximation theory approach [14], Bessel collocation method [31], Taylor matrix method [13], stochastic technique [3], and Chelyshkov collocation methods [4,27]. The other prominent aspects of the presented study is the development of two new operational matrices that are derived by using the modified GFDO of Caputo-type and GFIO of RL-type. In addition, according to our study, this is the first result, where numerical simulations of FDEs with modified GFDO of Caputo-type are executed by using operational matrices (OMs) of orthogonal polynomials.

The manuscript is organized as follows: The modified GFDO of Caputo-type and some of its useful properties are listed in Section 2. The representation of a square integrable function in terms of basis of the fractional Legendre vector function (LVF) is studied in Section 3. Section 4 is devoted to the development of operational matrices of GFIO and GFDO of fractional LVF. The outline of the presented numerical method is studied in Section 5. The applicability and accuracy of the presented method is tested and analyzed by solving various problems and comparing the obtained results to other existing numerical methods in Section 6. Finally, the proposed study is concluded in Section 7.

2. New modified GFDO of Caputo-type

The fractional-order operators in its generic form defined in (4), (5), and (6) are significantly influenced by the parameters, δ and η, therefore they have been extensively utilized in fractional-order modelling to explain various physical phenomena, see [1,8,9,11,12,16,32]. However, the GFDO defined in (5) and (6) do not satisfy a very useful generalized rule as in (2). So, in 2020, Odibat and Baleanu [22] presented the new modified GFDO of Caputo-type whose properties are somehow similar to the properties of Caputo operator, given in (1).

Definition 2.1: The new modified version of GFDO of Caputo-type of a function u is defined by (provided it exists)

$$ {}_{\mathrm{C}}\mathcal{D}_{a^+}^{\delta,\eta}u(z) = \frac{\eta^{\delta-n+1}}{\Gamma(n-\delta)}\int_a^z x^{\eta-1}(z^\eta - x^\eta)^{n-\delta-1}\left(x^{1-\eta}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n u(x)\,\mathrm{d}x, \quad z > a \geq 0,\ \eta > 0, \tag{7}$$

where $n-1 < \delta < n$ is the order of the modified GFDO.

We have the following useful observations for the modified GFDO of Caputo-type:

$$
{}_{C}\mathcal{D}^{\delta,\eta}_{a^+}\left(z^\eta - a^\eta\right)^\rho = \begin{cases} \eta^\delta \dfrac{\Gamma(\rho+1)}{\Gamma(\rho-\delta+1)}\left(z^\eta - a^\eta\right)^{\rho-\delta}, & \rho \in \mathbb{R}_+ \text{ and } \rho \geq \lceil \delta \rceil, \\ 0, & \rho \in \mathbb{R}_+ \text{ and } \rho < \lceil \delta \rceil. \end{cases} \tag{8}
$$

The modified GFDO of Caputo-type has linear operation

$$
{}_{C}\mathcal{D}^{\delta,\eta}_{a^+}\left(\sum_{j=1}^{n} a_j u_j(z)\right) = \sum_{j=1}^{n} a_j {}_{C}\mathcal{D}^{\delta,\eta}_{a^+} u_j(z). \tag{9}
$$

Definition 2.2: The following is the definition of the beta function which plays an important role in the computation of the fractional derivative of power functions:

$$
B(x,y) = \int_0^1 z^{x-1}(1-z)^{y-1}\,\mathrm{d}z, \quad \operatorname{Re}(x) > 0,\ \operatorname{Re}(y) > 0. \tag{10}
$$

3. Fractional LVF and its properties

In 2011, Rida and Yousef [24] derived fractional extensions of Legendre polynomials (LPs) by applying Rodrigues' formula. However, the proposed extension in [24] is not easy to use for computational purposes due to the complexity involved in its construction. So considering the computational difficulties, Kazem and coauthors in 2013 introduced the fractional LVF which are the natural extension of LPs (see [19]). In this section, some necessary properties of fractional LVF are recalled.

3.1. Properties of fractional LVF

The fractional LVF can be computed by using the following relation (see [19]). For easiness, the notation $FL^\beta_k(z)$ is used to indicate fractional LVF:

$$
\begin{aligned}
&FL^\beta_{k+1}(z) = \frac{(2k+1)(2z^\beta - 1)}{k+1} FL^\beta_k(z) - \frac{k}{k+1} FL^\beta_{k-1}(z), \quad k \in \mathbb{N}, \\
&FL^\beta_0(z) = 1, \quad FL^\beta_1(z) = 2z^\beta - 1.
\end{aligned} \tag{11}
$$

Equation (11) can also be written as

$$
\begin{aligned}
&FL^\beta_k(z) = \sum_{s=0}^{k} \Omega_{(s,k)} z^{s\beta}, \\
&\Omega_{(s,k)} = (-1)^{s+k} \frac{\Gamma(1+s+k)}{\Gamma(1-s+k)(\Gamma(1+s))^2}.
\end{aligned} \tag{12}
$$

The orthogonality conditions with respect to the weight function $w(z) = z^{\beta-1}$ are

$$
\int_0^1 FL^\beta_k(z) FL^\beta_{k'}(z) w(z)\,\mathrm{d}z = \begin{cases} \dfrac{1}{\beta(2k+1)}, & \text{for } k = k', \\ 0, & \text{for } k \neq k'. \end{cases} \tag{13}
$$

3.2. Function approximations

Any function $u \in L[0,1]$ can be expressed in terms of basis of the fractional LVF in the form

$$u(z) = \sum_{k=0}^{\infty} a_k FL_k^{\beta}(z). \tag{14}$$

Using (13), we can easily compute a_k as

$$a_k = \beta(2k+1)\int_0^1 u(z)FL_k^{\beta}(z)w(z)\,\mathrm{d}z, \quad k \in \mathbb{N}_0. \tag{15}$$

For fixed M, we may express (14) as

$$u(z) \simeq \sum_{k=0}^{M} a_k FL_k^{\beta}(z) = \Upsilon^T \Theta(u), \tag{16}$$

where

$$\Upsilon^T = [a_0, a_1, \ldots, a_M]$$

and

$$\Theta(z) = [FL_0^{\beta}(z), FL_1^{\beta}(z), FL_2^{\beta}(z), \ldots, FL_M^{\beta}(z)]^T. \tag{17}$$

4. Fractional LVF OMs of GFIO and modified GFDO

Lemma 4.1: *The modified generalized fractional-order derivative of $FL_k^{\beta}(z)$ defined in (12) can be computed by*

$$_{\mathrm{C}}\mathcal{D}_{0^+}^{\delta,\eta} FL_k^{\beta}(z) = \sum_{s=0}^{k} \Omega_{(s,k)}\eta^{\delta}\frac{\Gamma(s\beta/\eta+1)}{\Gamma(s\beta/\eta-\delta+1)}z^{\beta s-\eta\delta}, \quad \eta,\beta \in \mathbb{R}_+,\ \delta > 0. \tag{18}$$

Proof: Using (8), (9), and (12), the result can be proved. ■

Lemma 4.2: *The generalized fractional-order integral of the function $(z^{\eta}-a^{\eta})^{\rho}$ in RL-type can be computed by*

$$J_{a^+}^{\delta,\eta}\left(z^{\eta}-a^{\eta}\right)^{\rho} = \eta^{-\delta}\frac{\Gamma(\rho+1)}{\Gamma(\rho+\delta+1)}\left(z^{\eta}-a^{\eta}\right)^{\rho+\delta}, \quad z > a \geq 0,\ \eta \in \mathbb{R}_+,\ \delta > 0. \tag{19}$$

Proof: Using the substitution $\xi = \frac{x^{\eta}-a^{\eta}}{z^{\eta}-a^{\eta}}$ into (4), then using the definition of the beta function (10), the result can be proved. ■

Corollary 4.3: *For $\delta > 0$, $s \in \mathbb{N}$, and $\beta \in \mathbb{R}_+$, we have*

$$J_{0^+}^{\delta,\eta}(z)^{\beta s} = \eta^{-\delta}\frac{\Gamma(\beta s/\eta+1)}{\Gamma(\beta s/\eta+\delta+1)}z^{\beta s+\eta\delta}, \quad \eta > 0. \tag{20}$$

Lemma 4.4: *The generalized fractional-order integral in RL sense of $FL_k^{\beta}(z)$ defined in (12) can be computed by*

$$J_{0^+}^{\delta,\eta} FL_k^{\beta}(z) = \sum_{s=0}^{k} \Omega_{(s,k)}\eta^{-\delta}\frac{\Gamma(s\beta/\eta+1)}{\Gamma(s\beta/\eta+\delta+1)}z^{\beta s+\eta\delta}, \quad \eta \in \mathbb{R}_+. \tag{21}$$

Proof: Applying linearity of fractional-order integral of RL-type to (12), we have

$$J_{0^+}^{\delta,\eta} FL_k^{\beta}(z) = \sum_{s=0}^{k} \Omega_{(s,k)} J_{0^+}^{\delta,\eta} z^{\beta s}. \tag{22}$$

Using (20), the result can be proved. ■

Lemma 4.5: *For $\delta > 0$ and $\beta \in \mathbb{R}_+$, we have*

$$z^{s\beta+\delta\eta} \simeq \sum_{j=0}^{M} b_j FL_j^{\beta}(z), \quad \eta > 0,$$

$$b_j = \beta\left(2j+1\right) \sum_{r=0}^{j} \frac{(-1)^{(j+r)}\Gamma(j+r+1)}{\Gamma(j-r+1)(\Gamma(r+1))^2 \beta\,(s+r+1)+\delta\eta}, \quad j = 0, 1, \ldots, M.$$

Proof: Using $(M+1)$ terms of the fractional LVF, we may approximate $z^{s\beta+\delta\eta}$ as

$$z^{s\beta+\delta\eta} \simeq \sum_{j=0}^{M} b_j FL_j^{\beta}(z). \tag{23}$$

Using (15), we may determine b_j as

$$\begin{aligned} b_j &= \beta(2j+1) \int_0^1 FL_j^{\beta}(z) z^{s\beta+\delta\eta} z^{\beta-1}\, \mathrm{d}z \\ &= \beta(2j+1) \sum_{r=0}^{j} \frac{(-1)^{j+r}\Gamma(j+r+1)}{\Gamma(j-r+1)(\Gamma(r+1))^2} \int_0^1 z^{s\beta+\delta\eta+\beta-1+r\beta}\, \mathrm{d}z \\ &= \beta(2j+1) \sum_{r=0}^{j} \frac{(-1)^{j+r}\Gamma(j+r+1)}{\Gamma(j-r+1)(\Gamma(r+1))^2} \frac{1}{s\beta+\delta\eta+\beta+r\beta}. \end{aligned} \tag{24}$$

Equation (24) can also be expressed as

$$b_j = \beta\left(2j+1\right) \sum_{r=0}^{j} \frac{(-1)^{(j+r)}\Gamma(j+r+1)}{\Gamma(j-r+1)(\Gamma(r+1))^2 \beta\,(s+r+1)+\delta\eta}. \tag{25}$$

Equations (23) and (25) prove the result. ■

Lemma 4.6: *For $\delta > 0$ and $\beta \in \mathbb{R}_+$, we have*

$$z^{s\beta-\delta\eta} = \sum_{j=0}^{M} e'_j FL_j^{\beta}(z), \quad \eta > 0,$$

$$e'_j = \beta\left(2j+1\right) \sum_{r=0}^{j} \frac{(-1)^{(j+r)}\Gamma(j+r+1)}{\Gamma(j-r+1)(\Gamma(r+1))^2 \beta\,(s+1+r)-\delta\eta}, \quad j = 0, 1, \ldots, M.$$

Proof: Using $(M+1)$ terms of the fractional LVF, we may approximate $z^{s\beta-\delta\eta}$ as

$$z^{s\beta-\delta\eta} \simeq \sum_{j=0}^{M} e'_j FL_j^{\beta}(z). \tag{26}$$

Using (15), we may compute e'_j as

$$\begin{aligned} e'_j &= \beta(2j+1)\int_0^1 FL_j^{\beta}(z) z^{s\beta-\delta\eta} z^{\beta-1}\,\mathrm{d}z \\ &= \beta(2j+1)\sum_{r=0}^{j} \frac{(-1)^{j+r}\Gamma(j+r+1)}{\Gamma(j-r+1)(\Gamma(r+1))^2}\int_0^1 z^{s\beta-\delta\eta+\beta-1+r\beta}\,\mathrm{d}z \\ &= \beta(2j+1)\sum_{r=0}^{j} \frac{(-1)^{j+r}\Gamma(j+r+1)}{\Gamma(j-r+1)(\Gamma(r+1))^2}\frac{1}{s\beta-\delta\eta+\beta+r\beta}. \end{aligned} \tag{27}$$

Equation (27) can also be written as

$$e'_j = \beta\left(2j+1\right)\sum_{r=0}^{j}\frac{(-1)^{(j+r)}\Gamma(j+r+1)}{\Gamma(j-r+1)(\Gamma(r+1))^2\beta\left(s+1+r\right)-\delta\eta}. \tag{28}$$

Equations (26) and (28) prove the result. ∎

4.1. New generalized OMs of GFIO of RL-type and modified GFDOs of Caputo-type

This section deals with the construction of the OMs of fractional LVF in the sense of GFIO of RL-type and modified GFDO of Caputo type (Figures 1, 2, and 4).

Theorem 4.7: *If $\Theta(z)$ is a fractional LVF, then*

$$J_{0^+}^{\delta,\eta}\Theta(z) \simeq \mathbf{P}^{\delta,\eta}_{(M+1,M+1)}\Theta(z), \tag{29}$$

where $\mathbf{P}^{\delta,\eta}$ is the $(M+1)\times(M+1)$ operational matrix of fractional-order integration of order $\delta>0$ in GFIO sense defined as

$$\mathbf{P}^{\delta,\eta}_{(M+1,M+1)} = \sum_{s=0}^{i}\varpi_{i,j,s}, \quad i=0,1,\ldots,M,\ j=0,1,\ldots,M, \tag{30}$$

where

$$\varpi_{i,j,s} = \beta\left(2j+1\right)\sum_{r=0}^{j}\frac{(-1)^{i+s+j+r}\eta^{-\delta}\Gamma(i+s+1)\Gamma(j+r+1)\Gamma(\beta s/\eta+1)}{\Gamma(i-s+1)(\Gamma(s+1))^2\Gamma(j-r+1)\times(\Gamma(r+1))^2\Gamma(\beta s/\eta+\delta+1)\beta\left(s+r+1\right)+\delta\eta}. \tag{31}$$

Proof: By applying the GFIO defined in (4) to (12), we have

$$J_{0^+}^{\delta,\eta} FL_i^{\beta}(z) = \sum_{s=0}^{i} \Omega_{(s,i)} J_{0^+}^{\delta,\eta}(z^{s\beta}). \tag{32}$$

Using Corollary 4.3, (32) can be expressed as

$$J_{0^+}^{\delta,\eta} FL_i^{\beta}(z) = \sum_{s=0}^{i} \Omega_{(s,i)} \eta^{-\delta} \frac{\Gamma(\beta s/\eta + 1)}{\Gamma(\beta s/\eta + \delta + 1)} z^{\beta s + \delta\eta}. \tag{33}$$

Using $(M+1)$ terms of the fractional LVF, $z^{s\beta+\delta\eta}$ can be approximated as

$$z^{s\beta+\delta\eta} \simeq \sum_{j=0}^{M} b_j FL_j^{\beta}(z). \tag{34}$$

Using Lemma 4.5, (34) can be expressed as

$$z^{s\beta+\delta\eta} = \sum_{j=0}^{M} \left(\beta\left(2j+1\right) \sum_{r=0}^{j} \frac{(-1)^{(j+r)}\Gamma(j+r+1)}{\Gamma(j-r+1)(\Gamma(r+1))^2 \beta\,(s+r+1)+\delta\eta} \right) FL_j^{\beta}(z). \tag{35}$$

Using (12) and (35) in (33), we have

$$\begin{aligned} J_{0^+}^{\delta,\eta} FL_i^{\beta}(z) \simeq \sum_{s=0}^{i} \frac{(-1)^{i+s}\Gamma(i+s+1)}{\Gamma(i-s+1)(\Gamma(s+1))^2} \eta^{-\delta} \frac{\Gamma(\beta s/\eta+1)}{\Gamma(\beta s/\eta+\delta+1)} \\ \sum_{j=0}^{M} \left(\beta\left(2j+1\right) \sum_{r=0}^{j} \frac{(-1)^{(j+r)}\Gamma(j+r+1)}{\Gamma(j-r+1)(\Gamma(r+1))^2 \beta\,(s+r+1)+\delta\eta} \right) FL_j^{\beta}(z). \end{aligned} \tag{36}$$

Equation (36) can further be written as

$$\begin{aligned} J_{0^+}^{\delta,\eta} FL_i^{\beta}(z) &\simeq \sum_{j=0}^{M} \left(\sum_{s=0}^{i} \beta\left(2j+1\right) \sum_{r=0}^{j} \frac{(-1)^{i+s+j+r}\eta^{-\delta}\Gamma(i+s+1)\Gamma(j+r+1)}{\Gamma(i-s+1)(\Gamma(s+1))^2\Gamma(j-r+1)(\Gamma(r+1))^2} \right. \\ &\qquad \left. \times \frac{\Gamma(\beta s/\eta+1)}{\Gamma(\beta s/\eta+\delta+1)\beta\,(s+r+1)+\delta\eta} \right) FL_j^{\beta}(z) \\ &= \sum_{j=0}^{M} \left(\sum_{s=0}^{i} \varpi_{i,j,s} \right) FL_j^{\beta}(z), \quad i,j = 0,1,\ldots,M, \end{aligned} \tag{37}$$

where $\varpi_{i,j,s}$ is given in (31). Now (37) in vector form can be written as

$$J_{0^+}^{\delta,\eta} FL_i^{\beta}(z) \simeq \left[\sum_{s=0}^{i} \varpi_{i,0,s}, \sum_{s=0}^{i} \varpi_{i,1,s}, \sum_{s=0}^{i} \varpi_{i,2,s} \cdots, \sum_{s=0}^{i} \varpi_{i,M,s} \right] \Theta(z). \tag{38}$$

Hence the result is proved. ■

Remark 4.8: The fractional-order integration operational matrix derived in [2] is a particular case of our derived operational matrix in Theorem 4.7 for $\beta = \eta = 1$.

Theorem 4.9: *If $\Theta(z)$ is a fractional LFV, then*

$$ {}_{C}\mathcal{D}_{0^+}^{\delta,\eta}\Theta(z) \simeq \mathbf{H}_{(M+1,M+1)}^{\delta,\eta}\Theta(z), \tag{39} $$

where $\mathbf{H}_{(M+1,M+1)}^{\delta,\eta}$ is the $(M+1)\times(M+1)$ operational matrix of fractional-order derivatives of order $\delta > 0$ in modified GFDO sense defined as

$$ \mathbf{H}_{(M+1,M+1)}^{\delta,\eta} = \sum_{s=\lceil \delta\eta/\beta \rceil}^{i} \omega_{(i,j,s)}, \quad i = \lceil \delta\eta/\beta \rceil, \dots, M,\ j = 0, \dots, M, \tag{40} $$

where

$$ \omega_{i,j,s} = \beta\left(2j+1\right)\sum_{r=0}^{j} \frac{(-1)^{i+s+j+r}\eta^{\delta}\Gamma(i+s+1)\Gamma(j+r+1)\Gamma(\beta s/\eta+1)}{\Gamma(i-s+1)(\Gamma(s+1))^2\Gamma(j-r+1)(\Gamma(r+1))^2 \times\Gamma(\beta s/\eta-\delta+1)\beta\left(s+r+1\right)-\delta\eta}. \tag{41} $$

Proof: By applying the modified GFDO defined in (7) to (12), we have

$$ {}_{C}\mathcal{D}_{0^+}^{\delta,\eta} FL_i^{\beta}(z) = \sum_{s=0}^{i} \Omega_{(s,i)}\, {}_{C}\mathcal{D}_{0^+}^{\delta,\eta}(z^{s\beta}). \tag{42} $$

Using Lemma 4.1, (42) can be expressed as

$$ {}_{C}\mathcal{D}_{0^+}^{\delta,\eta} FL_i^{\beta}(z) = \sum_{s=\lceil \delta\eta/\beta \rceil}^{i} \Omega_{(s,i)}\eta^{\delta}\frac{\Gamma(s\beta/\eta+1)}{\Gamma(s\beta/\eta-\delta+1)} z^{s\beta-\delta\eta}. \tag{43} $$

Using $(M+1)$ terms of the fractional LVF, $z^{s\beta-\delta\eta}$ can be approximated as

$$ z^{s\beta-\delta\eta} \simeq \sum_{j=0}^{M} e'_j FL_j^{\beta}(z). \tag{44} $$

Using Lemma 4.6, (44) can be expressed as

$$ z^{s\beta-\delta\eta} = \sum_{j=0}^{M}\left(\beta\left(2j+1\right)\sum_{r=0}^{j}\frac{(-1)^{(j+r)}\Gamma(j+r+1)}{\Gamma(j-r+1)(\Gamma(r+1))^2\beta\left(s+r+1\right)-\delta\eta}\right) FL_j^{\beta}(z). \tag{45} $$

Using (12) and (45) in (43), we have

$$ \begin{aligned} {}_{C}\mathcal{D}_{0^+}^{\delta,\eta} FL_i^{\beta}(z) \simeq{}& \sum_{s=\lceil \delta\eta/\beta \rceil}^{i} \frac{(-1)^{i+s}\Gamma(i+s+1)}{\Gamma(i-s+1)(\Gamma(s+1))^2}\eta^{\delta}\frac{\Gamma(\beta s/\eta+1)}{\Gamma(\beta s/\eta-\delta+1)} \\ & \sum_{j=0}^{M}\left(\beta\left(2j+1\right)\sum_{r=0}^{j}\frac{(-1)^{(j+r)}\Gamma(j+r+1)}{\Gamma(j-r+1)(\Gamma(r+1))^2\beta\left(s+r+1\right)-\delta\eta}\right) FL_j^{\beta}(z). \end{aligned} \tag{46} $$

Equation (46) can further be written as

$$ {}_{C}\mathcal{D}_{0^+}^{\delta,\eta} FL_i^{\beta}(z) \simeq \sum_{j=0}^{M}\left(\sum_{s=\lceil \delta\eta/\beta \rceil}^{i}\beta\left(2j+1\right)\sum_{r=0}^{j}\frac{(-1)^{i+s+j+r}\eta^{\delta}\Gamma(i+s+1)\Gamma(j+r+1)}{\Gamma(i-s+1)(\Gamma(s+1))^2\Gamma(j-r+1)(\Gamma(r+1))^2}\right. $$

$$
\times \frac{\Gamma(\beta s/\eta + 1)}{\Gamma(\beta s/\eta - \delta + 1)\beta\,(s + r + 1) - \delta\eta}\Bigg) FL_j^{\beta}(z)
$$

$$
= \sum_{j=0}^{M} \left(\sum_{s=\lceil \delta\eta/\beta \rceil}^{i} \omega_{i,j,s} \right) FL_j^{\beta}(z), \quad i = \lceil \delta\eta/\beta \rceil, \ldots, M,\ j = 0, 1, \ldots, M, \tag{47}
$$

where $\omega_{i,j,s}$ is given in (41). Now (47) in vector form can be written as

$$
{}_{C}\mathcal{D}_{0^+}^{\delta,\eta} FL_i^{\beta}(z) \simeq \left[\sum_{s=\lceil \delta\eta/\beta \rceil}^{i} \omega_{i,0,s}, \sum_{s=\lceil \delta\eta/\beta \rceil}^{i} \omega_{i,1,s}, \sum_{s=\lceil \delta\eta/\beta \rceil}^{i} \omega_{i,2,s}, \ldots, \sum_{s=\lceil \delta\eta/\beta \rceil}^{i} \omega_{i,M,s} \right] \Theta(z). \tag{48}
$$

Hence the result is proved. ∎

Remark 4.10: The fractional-order derivative operational matrix derived in [25] is a particular case of our derived operational matrix in Theorem 4.9 for $\beta = \eta = 1$.

Remark 4.11: The fractional-order derivative operational matrix derived in [19] is a particular case of our derived operational matrix in Theorem 4.9 for $\eta = 1$.

5. Outline of the method

In this section, the applicability of the newly derived generalized OMs is ensured by constructing a numerical method that is entirely dependent on the OMs.

Consider the following generalized form of FDEs corresponding to the initial conditions (ICs):

$$
{}_{C}\mathcal{D}^{\delta,\eta} u(z) = f\left(z, u(z), {}_{C}\mathcal{D}^{\delta_1,\eta_1} u(z), {}_{C}\mathcal{D}^{\delta_2,\eta_2} u(z), \ldots, {}_{C}\mathcal{D}^{\delta_n,\eta_n} u(z)\right), \ u^{(k)}(0) = g_k, \quad k = 0, 1, \ldots, n. \tag{49}
$$

Consider the approximation

$$
{}_{C}\mathcal{D}^{\delta,\eta} u(z) = \Upsilon^T \Theta(z). \tag{50}
$$

Applying the GFIO of order δ to (50) and the initial conditions defined in (49), we have

$$
u(z) = \Upsilon^T J^{\delta,\eta} \Theta(z) + \sum_{k=0}^{n} g_k z^k. \tag{51}
$$

In the light of (29), we can write (51) as

$$
u(z) \simeq \Upsilon^T \mathbf{P}_{(M+1,M+1)}^{\delta,\eta} \Theta(z) + \sum_{k=0}^{n} g_k z^k. \tag{52}
$$

Approximating the series terms $\sum_{k=0}^{n} g_k z^k$ by using fractional LVF, we may write (52) as

$$
u(z) = \Upsilon^T \mathbf{P}_{(M+1,M+1)}^{\delta,\eta} \Theta(z) + G^T \Theta(z). \tag{53}
$$

The derivative terms of (49) can be approximated by using (39) and (53) as

$$\begin{cases} {}_C\mathcal{D}^{\delta_1,\eta_1}u(z) = \Upsilon^T \mathbf{P}^{\delta,\eta}_{(M+1,M+1)}\mathbf{H}^{\delta_1,\eta_1}_{(M+1,M+1)}\Theta(z) + G^T\mathbf{H}^{\delta_1,\eta_1}_{(M+1,M+1)}\Theta(z), \\ {}_C\mathcal{D}^{\delta_2,\eta_2}u(z) = \Upsilon^T \mathbf{P}^{\delta,\eta}_{(M+1,M+1)}\mathbf{H}^{\delta_2,\eta_2}_{(M+1,M+1)}\Theta(z) + G^T\mathbf{H}^{\delta_2,\eta_2}_{(M+1,M+1)}\Theta(z), \\ \vdots \quad \vdots \quad \vdots \\ {}_C\mathcal{D}^{\delta_n,\eta_n}u(z) = \Upsilon^T \mathbf{P}^{\delta,\eta}_{(M+1,M+1)}\mathbf{H}^{\delta_n,\eta_n}_{(M+1,M+1)}\Theta(z) + G^T\mathbf{H}^{\delta_n,\eta_n}_{(M+1,M+1)}\Theta(z), \\ \text{and} \\ f(z) = B^T\Theta(z). \end{cases} \tag{54}$$

Now putting (50) and (54) in (49), we have

$$\begin{aligned} \Upsilon^T\Theta(z) &= \Upsilon^T\mathbf{P}^{\delta,\eta}_{(M+1,M+1)}\left(\mathbf{H}^{\delta_1,\eta_1}_{(M+1,M+1)} + \mathbf{H}^{\delta_2,\eta_2}_{(M+1,M+1)} + \cdots + \mathbf{H}^{\delta_n,\eta_n}_{(M+1,M+1)}\right)\Theta(z) \\ &\quad + G^T\left(\mathbf{H}^{\delta_1,\eta_1}_{(M+1,M+1)} + \mathbf{H}^{\delta_2,\eta_2}_{(M+1,M+1)} + \cdots + \mathbf{H}^{\delta_n,\eta_n}_{(M+1,M+1)}\right)\Theta(z) + B^T\Theta(z). \end{aligned} \tag{55}$$

For computational purposes, we may write (55) in a simplified form as

$$\Upsilon^T - \Upsilon^T\mathbf{P}^{\delta,\eta}_{(M+1,M+1)}\widehat{H} = G^T\widehat{H} + B^T, \tag{56}$$

where $\widehat{H} = (\mathbf{H}^{\delta_1,\eta_1}_{(M+1,M+1)} + \mathbf{H}^{\delta_2,\eta_2}_{(M+1,M+1)} + \cdots + \mathbf{H}^{\delta_n,\eta_n}_{(M+1,M+1)})$. Equation (56) is a Sylvester-type matrix equation. The approximate solution of the generalized fractional problem (49) can be determined by putting the values of the unknown vector Υ^T in (53).

6. Applications of the method

In this section, we solve various fractional-order problems that include the modified GFDO to determine the accuracy and efficiency of our presented method (PM). The accuracy of PM is also highlighted by comparing the results obtained otherwise in the literature using various numerical methods.

Example 6.1: Consider the problem in modified GFDO of Caputo-type [14]

$$\begin{aligned} {}_C\mathcal{D}^{\delta,\eta}u(z) &= a_1{}_C\mathcal{D}^{\delta_1,\eta_1}u(z) + a_2{}_C\mathcal{D}^{\delta_2,\eta_2}u(z) + a_3{}_C\mathcal{D}^{\delta_3,\eta_3}u(z) \\ &\quad + a_4{}_C\mathcal{D}^{\delta_4,\eta_4}u(z) + F(z), \quad z \in [0,1],\ 0 < \delta < 2, \\ &u(0) = 0,\ u'(0) = 0. \end{aligned} \tag{57}$$

The source term is

$$F(z) = 4\,z - z^2 - \frac{6776\,z^{\frac{3}{2}}}{4503} + 42\,z^5 - 14\,z^6 + z^7 + \frac{1516\,z^{\frac{13}{2}}}{5629} - 2.$$

The exact solution at $\delta = 2, a_1 = a_3 = -1, a_2 = 2, a_4 = 0, \delta_1 = 0, \delta_2 = 1, \delta_3 = \frac{1}{2}, \eta = \eta_1 = \eta_2 = \eta_3 = \eta_4 = 1$ is

$$u(z) = z^7 - z^2.$$

We now test the applicability and numerical efficiency of our PM. The results are studied for various values of M, β, and δ. We see that as the number of terms of the fractional LVF is increased, the absolute error between the approximate solution and the exact solution is reduced, see Table 1. Also, by

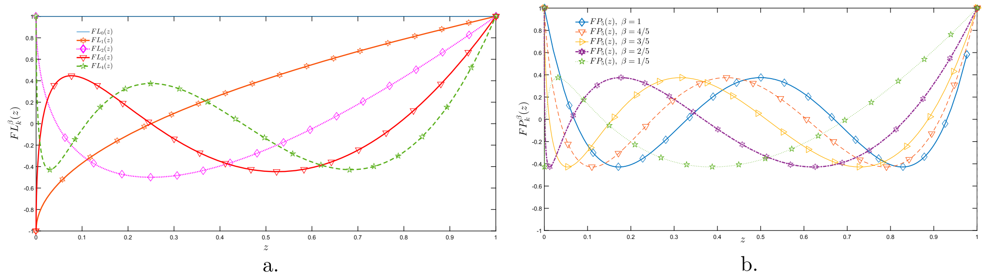

Figure 1. (a) Fractional LVF at $\beta = 0.5$ and various choices of k. (b) Fractional LVF at $k = 4$ and various choices of β.

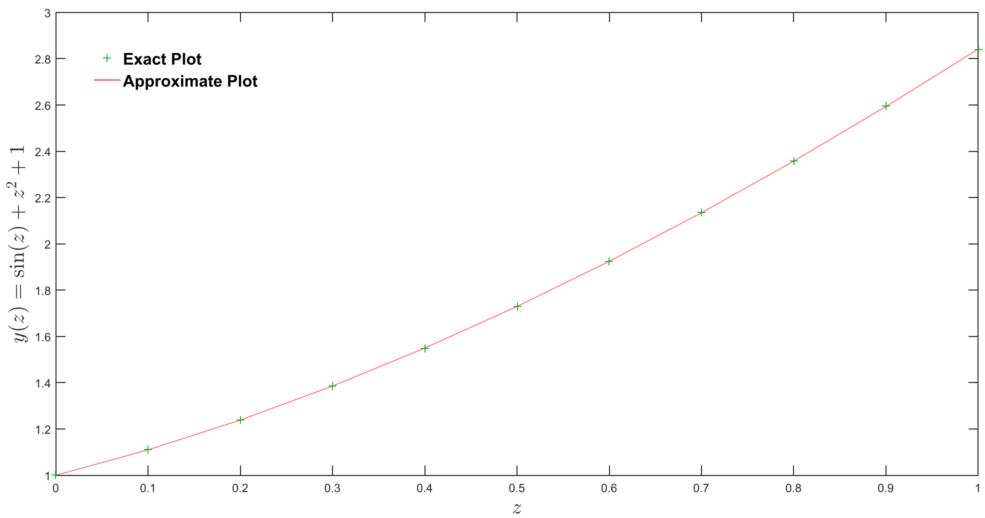

Figure 2. Exact and approximate plots at $\beta = 0.5$ and $M = 4$.

Table 1. Comparison of absolute errors of Example 6.1 at $\beta = \eta = 1$ and various values of M.

z	$M = 4$ [14]	$M = 4$ PM	$M = 6$ [14]	$M = 6$ PM	$M = 7$ [14]	$M = 7$ PM
0.2	8.44×10^{-2}	1.32×10^{-2}	4.4×10^{-3}	3.0×10^{-4}	$2.81025203108243 \times 10^{-15}$	0
0.4	3.501×10^{-2}	1.05×10^{-2}	7.9×10^{-3}	3.5×10^{-4}	$6.63358257213531 \times 10^{-15}$	0
0.6	6.734×10^{-2}	3.49×10^{-2}	1.43×10^{-2}	3.4×10^{-4}	$3.27515792264421 \times 10^{-15}$	0
0.8	1.0234	3.87×10^{-2}	2.14×10^{-2}	6.9×10^{-4}	$4.25770529943748 \times 10^{-14}$	0
1	1.6700	7.8×10^{-2}	2.80×10^{-2}	1.03×10^{-3}	$2.43819897540083 \times 10^{-13}$	0

increasing the values of M, the accuracy in the approximate results is increased, see Figure 3. We also compute the absolute errors at various fractional values of β and obtain promising results, see Table 2. The impact of the fractional parameter δ is also analyzed by computing the approximate solution at its various values, see Figure 4. We observe that as $\delta \to 2$, the approximate solution approaches the exact solution of the problem (57). The effect of the real number β is also examined by computing the approximate solution at its various values and observing a great resemblance between the exact and approximate solutions, see Figure 5. Moreover, the results determined otherwise in [14] are also compared with the results obtained by using our PM. We observe that our results are more promising for this particular problem, see Table 1.

Example 6.2: Consider the problem in modified GFDO of Caputo-type [25]

$$ {}_C\mathcal{D}^{\delta,\eta}u(z) = -u(z),\ z \in [0,1],\ 0 < \delta < 2, $$

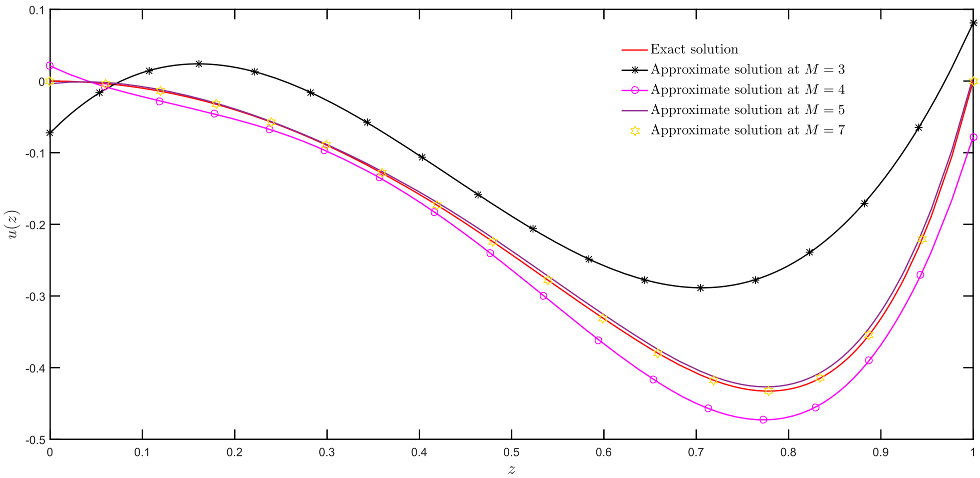

Figure 3. Approximate plots of Example 6.1 at various values of M.

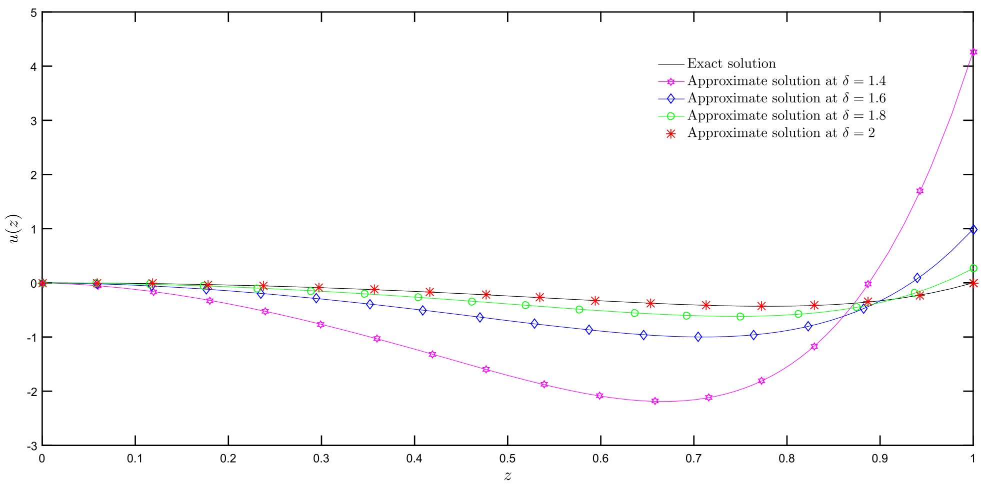

Figure 4. Approximate plots of Example 6.1 at various values of δ.

Table 2. Absolute errors computed for Example 6.1 at $M = 10$ and various values of β.

z	$\beta = 0.7$	$\beta = 0.8$	$\beta = 0.9$	$\beta = 1.8$	$\beta = 1.9$	$\beta = 2$
0.2	2.5×10^{-5}	1.3×10^{-4}	1.7×10^{-4}	1.3×10^{-4}	9.1×10^{-5}	4.03×10^{-6}
0.4	9.2×10^{-5}	5.2×10^{-4}	7.2×10^{-4}	3.9×10^{-4}	2.5×10^{-4}	9.13×10^{-6}
0.6	2.04×10^{-5}	1.2×10^{-3}	1.8×10^{-3}	6.3×10^{-4}	3.9×10^{-4}	1.54×10^{-5}
0.8	3.6×10^{-4}	2.2×10^{-3}	3.2×10^{-3}	8.9×10^{-4}	5.5×10^{-4}	2.23×10^{-5}
1	5.8×10^{-4}	3.5×10^{-3}	5.4×10^{-3}	1.2×10^{-3}	7.4×10^{-4}	3.10×10^{-5}

$$u(0) = 1, \; u'(0) = 0. \tag{58}$$

If we choose $\delta > 1$, then the condition $u'(0) = 0$ is applicable. The exact solution is

$$u(z) = \sum_{k=0}^{\infty} \frac{\left(-z^{\delta}\right)^k}{\Gamma(\delta k + 1)}.$$

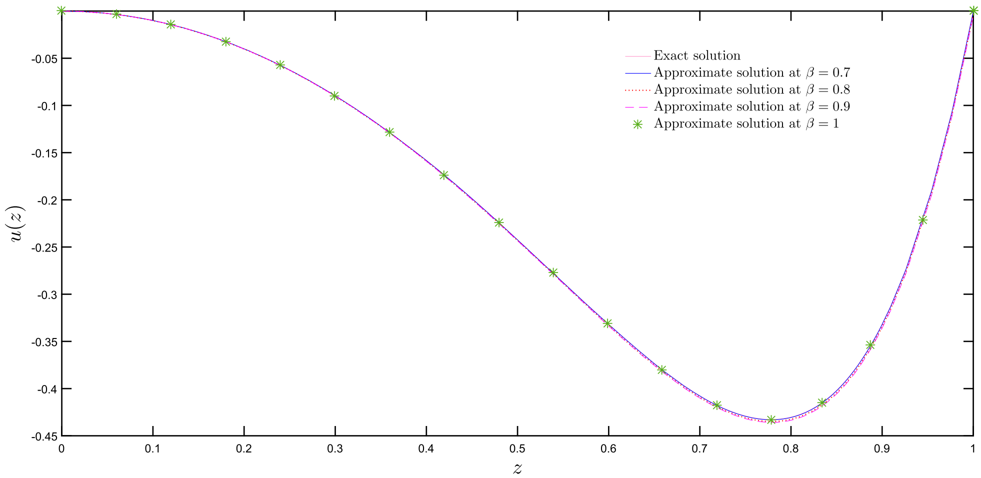

Figure 5. Approximate plots of Example 6.1 at $M = 10$ and various values of β.

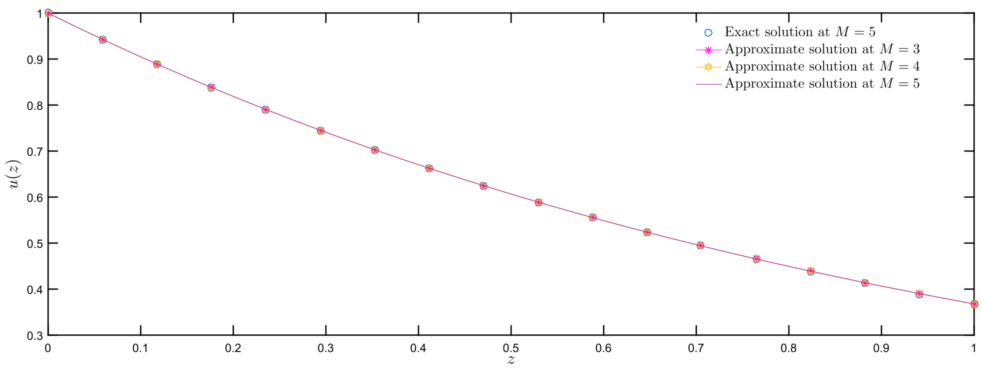

Figure 6. Approximate plots of Example 6.2 at various values of M.

For $\delta = 1 = \eta$, the exact solution is given as $u(z) = \exp(-z)$, and for $\delta = 2, \eta = 1$, the exact solution is given as $u(z) = \cos(z)$. We solve (58) for various values of M, β, η, and δ. In Table 3, we highlight the numerical efficiency of our PM by doing the comparison between the results computed by using our PM and the method presented in [25]. We also examine the applicability of the method by computing the approximate solution at various values of M and the fractional parameter δ. We observe that as M is increased, the accuracy in the approximate results is also increased, see Figure 6. Moreover, as $\delta \to 1$, the approximate solution approaches the exact solution of the problem (58), see Figure 7. We also determine the effects of β at various $\beta = \delta$ by computing the absolute errors, see Table 4. The impact of the parameter η is also analyzed by computing the approximate solution at its various values and obtaining promising results, see Figures 9 and 10. We also compute the absolute errors at various values of M and observe that the amount of absolute errors is decreasing between exact and approximate solutions with the increase in the values of M, see Figure 8.

Example 6.3: Consider the Bagley–Torvik problem in modified GFDO of Caputo-type [3,13,19,31]

$$\begin{aligned} &{}_{\mathrm{C}}\mathcal{D}^{\delta,\eta}u(z) = a_1 {}_{\mathrm{C}}\mathcal{D}^{\delta_1,\eta_1}u(z) - a_2 u(z) + F(z), \quad z \in [0,1],\ 0 < \delta < 2, \\ &u(0) = 1 = u'(0). \end{aligned} \tag{59}$$

Table 3. Comparison of absolute errors for Example 6.2 computed by using our PM and the method in [25] at $M = 10$, $\beta = \eta = 1$, and various values of δ.

δ	$z = 0.1$	$z = 0.1PM$	$z = 0.3$	$z = 0.3PM$	$z = 0.5$	$z = 0.5PM$	$z = 0.7$	$z = 0.7PM$	$z = 0.9$	$z = 0.9PM$
1.2	3.1×10^{-3}	2.2×10^{-5}	2.8×10^{-3}	1.9×10^{-5}	4.5×10^{-3}	1.2×10^{-3}	3.6×10^{-3}	1.6×10^{-5}	1.8×10^{-3}	2.0×10^{-3}
1.4	1.0×10^{-3}	1.6×10^{-5}	7.0×10^{-4}	1.4×10^{-5}	1.3×10^{-3}	7.9×10^{-6}	1.1×10^{-3}	1.1×10^{-5}	2.4×10^{-4}	1.4×10^{-5}
1.6	3.0×10^{-4}	7.5×10^{-6}	1.3×10^{-4}	6.3×10^{-6}	3.1×10^{-4}	3.4×10^{-6}	3.0×10^{-4}	4.9×10^{-6}	$6.2e \times 10^{-7}$	6.1×10^{-6}
1.8	6.1×10^{-5}	2.2×10^{-6}	1.4×10^{-5}	1.8×10^{-6}	4.9×10^{-5}	9.2×10^{-7}	5.3×10^{-5}	1.4×10^{-6}	8.8×10^{-6}	1.7×10^{-6}
0.2	2.9×10^{-1}	2.3×10^{-3}	4.5×10^{-1}	1.9×10^{-2}	$7.4e \times 10^{-1}$	5.3×10^{-2}	3.7×10^{-3}	1.0×10^{-1}	2.0×10^{-1}	1.8×10^{-1}
0.4	3.9×10^{-1}	7.0×10^{-4}	5.1×10^{-1}	1.0×10^{-3}	$7.3e \times 10^{-1}$	1.5×10^{-3}	3.3×10^{-1}	2.4×10^{-3}	2.2×10^{-1}	8.2×10^{-3}
0.6	6.7×10^{-3}	3.8×10^{-4}	2.0×10^{-5}	4.4×10^{-4}	$5.2e \times 10^{-3}$	3.3×10^{-4}	4.4×10^{-3}	3.42×10^{-4}	4.6×10^{-3}	3.4×10^{-4}
0.8	1.1×10^{-3}	1.1×10^{-4}	2.1×10^{-4}	1.1×10^{-4}	8.4×10^{-4}	7.8×10^{-5}	8.7×10^{-4}	9.5×10^{-5}	5.8×10^{-4}	1.2×10^{-4}

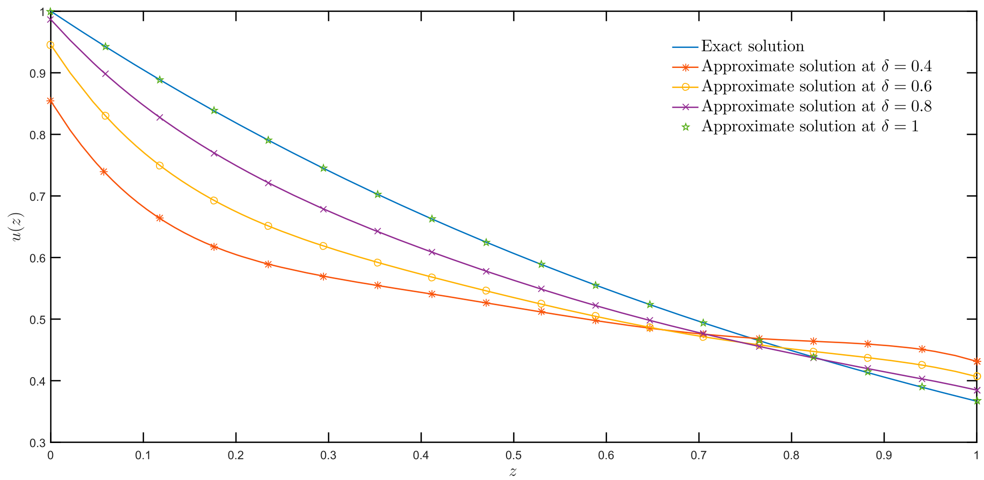

Figure 7. Approximate plots of Example 6.2 at various values of δ.

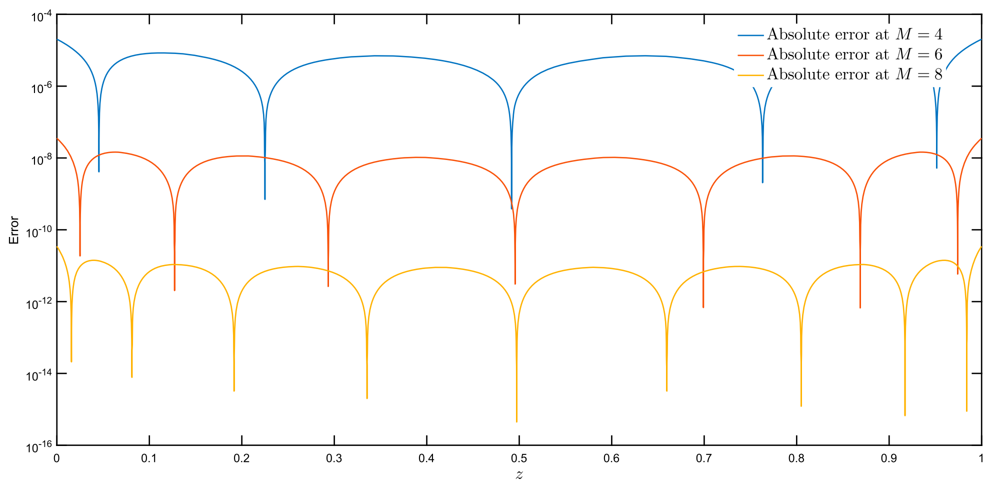

Figure 8. Error plots of Example 6.2 at various values of M.

Table 4. Absolute errors computed for Example 6.2 at $M = 10$ and various values of $\beta = \delta$.

z	$\beta = \delta = 0.5$	$\beta = \delta = 0.75$	$\beta = \delta = 0.67$	$\beta = \delta = 1.25$	$\beta = \delta = 1.75$	$\beta = \delta = 1.5$
0.2	4.21×10^{-7}	2.51×10^{-11}	6.98×10^{-10}	8.74×10^{-17}	2.67×10^{-18}	1.18×10^{-17}
0.4	1.80×10^{-5}	6.88×10^{-9}	1.07×10^{-7}	4.16×10^{-16}	9.56×10^{-19}	1.46×10^{-16}
0.6	1.60×10^{-4}	1.90×10^{-7}	2.02×10^{-6}	2.01×10^{-14}	3.26×10^{-18}	3.28×10^{-16}
0.8	7.48×10^{-4}	1.97×10^{-6}	1.61×10^{-5}	1.01×10^{-12}	4.16×10^{-18}	7.98×10^{-16}
1	2.47×10^{-3}	1.21×10^{-5}	8.05×10^{-5}	2.16×10^{-11}	2.95×10^{-17}	1.22×10^{-14}

The source term is

$$F(z) = 1 + z.$$

The exact solution of (59) at $\delta = 2$, $\delta_1 = 1.5$, $a_1 = -1 = a_2$, and $\eta = \eta_1 = 1$ is

$$u(z) = 1 + z. \tag{60}$$

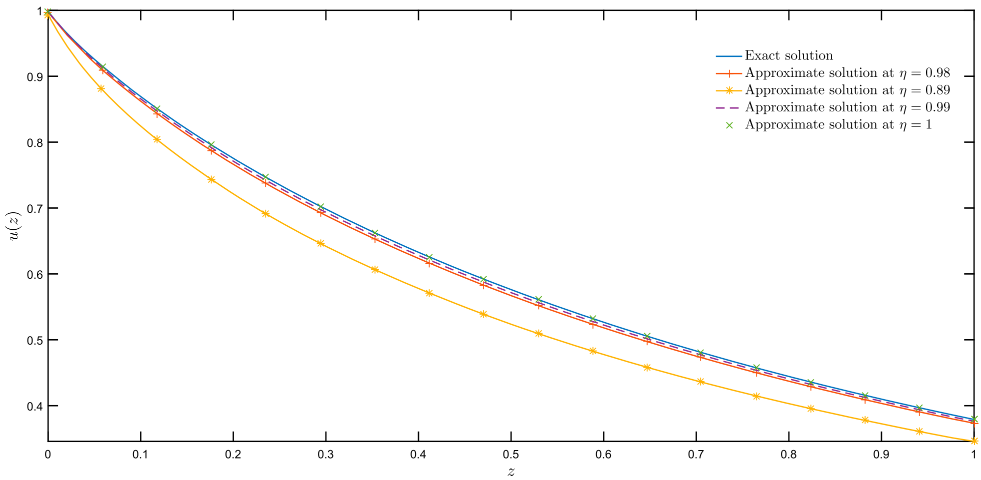

Figure 9. Approximate plots of Example 6.2 at $M = 10$, $\delta = 0.87$, and various values of η.

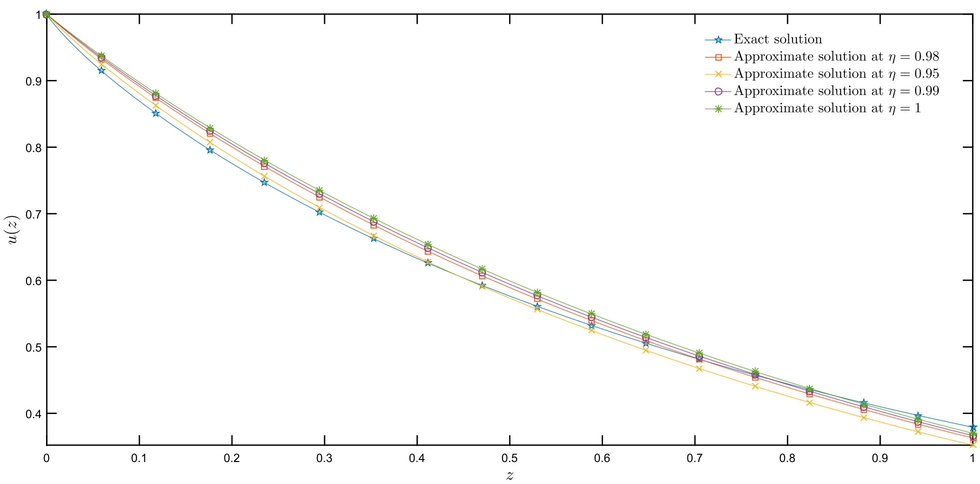

Figure 10. Approximate plots of Example 6.2 at $M = 10$, $\delta = 0.97$, and various values of η.

We solve (59) by using our PM and compare the results with the methods proposed in [3,13,31] to demonstrate the applicability and numerical efficiency of our method. We observe that our PM exhibits more accurate results with high precision in the numerical solutions, see Tables 5, 6, and Figure 11. Even at low values of M, we achieve promising results. The applicability of the parameters β and δ is also examined by computing the approximate solution at their various noninteger values. We observe that even at a very low value of M, the exact solution of the problem (59) is obtained, see Figure 12. We also examine the effects of the real number β for various $\delta = \beta$ at $M = 2$ and observe the great resemblance of the approximate solution with the exact solution, see Figure 13. The numerical efficiency is also highlighted by computing the approximate solution at various values of M, it is noted that with the increase in the values of M, the approximate solution matches exactly with the exact solution of (59), see Figure 13 and Table 5.

Example 6.4: Consider the problem in modified GFDO of Caputo-type [14]

$$ {}_{C}\mathcal{D}^{\delta,\eta}u(z) = a_1 {}_{C}\mathcal{D}^{\delta_1,\eta_1}u(z) + a_2 {}_{C}\mathcal{D}^{\delta_2,\eta_2}u(z) + a_3 {}_{C}\mathcal{D}^{\delta_3,\eta_3}u(z) $$

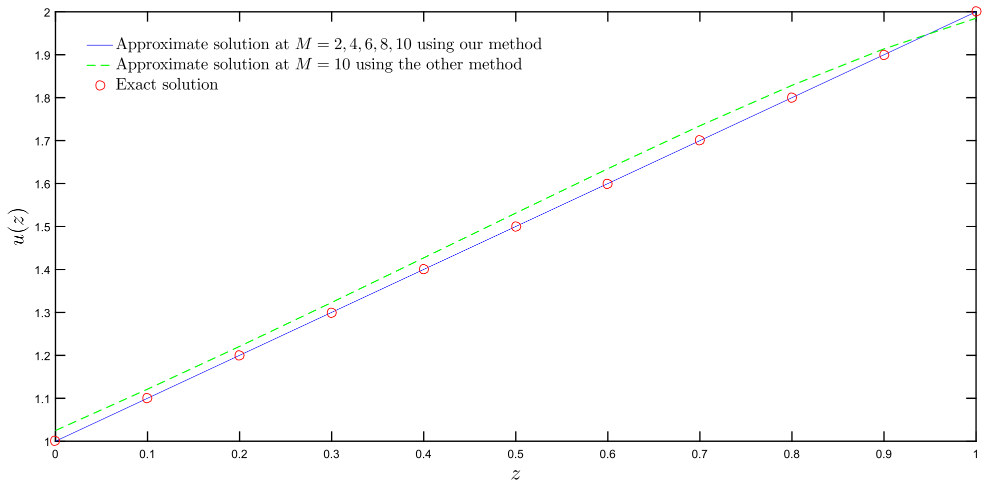

Figure 11. Approximate results of Example 6.3 are compared with the results of [3, Example 2].

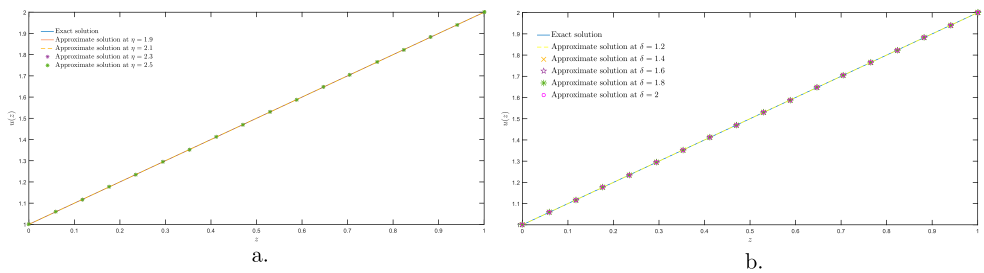

Figure 12. (a) Approximate plots of Example 6.3 at $M = 2$, $\beta = 1$, and various values of η. (b) Approximate plots of Example 6.3 at $M = 2$, $\beta = 1 = \eta$, and various values of δ.

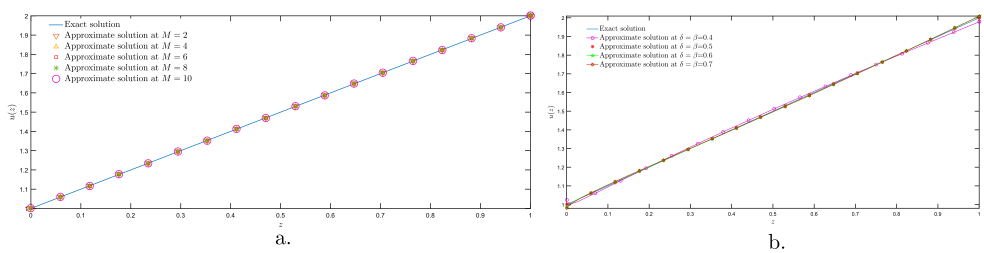

Figure 13. (a) Approximate plots of Example 6.3 at $\beta = \eta = 1$ and various values of M. (b) Approximate plots of Example 6.3 at $M = 2$, $\eta = 1$, and various values of $\delta = \beta$.

$$
\begin{aligned}
&+ a_{4C}\mathcal{D}^{\delta_4,\eta_4}u(z) + F(z),\ z \in [0,1],\ 0 < \delta < 2, \\
&u(0) = 0,\ u'(0) = 0.
\end{aligned}
\tag{61}
$$

The source term is

$$
F(z) = 6z + z^3 - \frac{12}{\Gamma\left(\frac{7}{3}\right)} z^{\frac{4}{3}} + \frac{6}{\Gamma\left(\frac{10}{3}\right)} z^{\frac{7}{3}}.
$$

Table 5. Comparison of absolute errors for Example 6.3 at $\eta = 1, \beta = 1$, and various values of M.

z	$M = 10$ using [3]	$M = 2, 4, 6, 8, 10$ using PM
0	2.30×10^{-2}	0
0.1	2.69×10^{-2}	0
0.2	3.13×10^{-2}	0
0.3	3.45×10^{-2}	0
0.4	3.45×10^{-2}	0
0.5	2.87×10^{-2}	0
0.6	1.36×10^{-2}	0
0.7	1.49×10^{-2}	0
0.8	2.30×10^{-2}	0
0.9	2.69×10^{-2}	0
1.0	3.13×10^{-2}	0

Table 6. Comparison of approximate solutions for Example 6.3.

z	Exact solution	PM at $M = 2$	Method in [13] at $M = 6$	Method in [31] at $M = 6$
0.2	1.20	1.20	1.20	1.20
0.4	1.40	1.40	1.40	1.40
0.6	1.60	1.60	1.60	1.60
0.8	1.80	1.80	1.80	1.80
1.0	2.00	2.00	2.00	2.00

The exact solution of (61) at $\delta = 2, \eta_1 = 1, a_1 = a_3 = -1, a_2 = 0, a_4 = 2, \delta_1 = 0, \delta_2 = \frac{2}{3}$, and $\delta_3 = \frac{5}{3}$ is

$$u(z) = z^3.$$

Here for $M = 3$, we have

$$\begin{aligned} \Theta(z) &= \begin{pmatrix} 1 \\ 2z-1 \\ 6z^2-6z+1 \\ 20z^3-30z^2+12z-1 \end{pmatrix}, \ \Upsilon = \begin{pmatrix} 3.0000000 \\ 3.0000000 \\ 0.0000000 \\ -0.0000000 \end{pmatrix}, \ G = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\ P^{2,1}_{(3+1,3+1)} &= \begin{pmatrix} 0.1666667 & 0.2500000 & 0.0833333 & 0 \\ -0.0833333 & -0.1000000 & 0 & 0.0166667 \\ 0.0166667 & 0 & -0.0238095 & 0 \\ 0 & 0.0071429 & 0 & -0.0111111 \end{pmatrix}. \end{aligned} \tag{62}$$

By putting the values of (62) into (53), the exact solution $u(z) = z^3$ of (61) is obtained. We analyse the applicability and the numerical efficiency of our PM by solving (61) at various values of M, and comparing the results with the results obtained in [14]. We observe that as M increased, the proposed method produces efficient numerical results, see Figure 14 and Table 7.

Example 6.5: Consider the problem in modified GFDO of Caputo-type [19]

$$\begin{aligned} {}_C\mathcal{D}^{\delta,\eta}u(z) - u(z) &= 1, \ z \in [0,1], \ 0 < \delta \leq 1, \\ u(0) &= 0. \end{aligned} \tag{63}$$

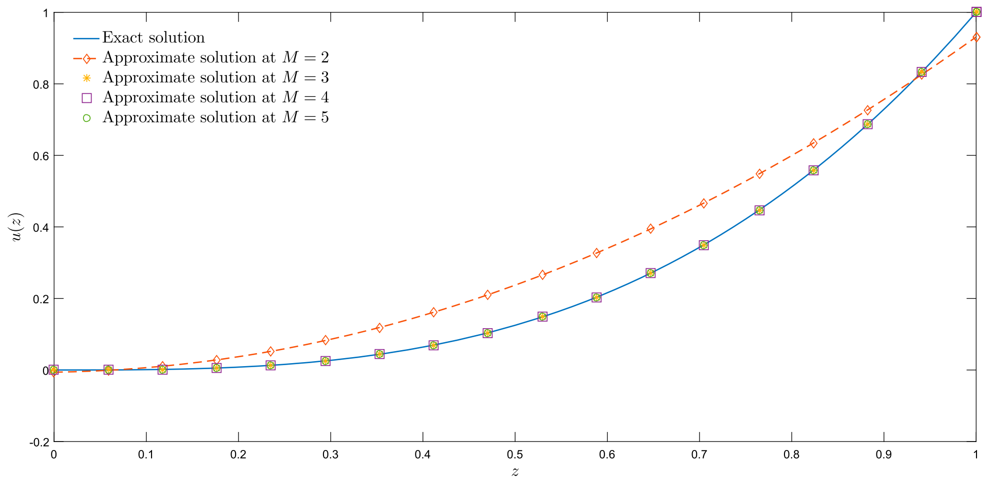

Figure 14. Approximate and exact plots of Example 6.4 at $\delta = 2, \eta = 1, \beta = 1$, and various values of M.

Table 7. Comparison of absolute errors of Example 6.4 by using PM and the method in [14].

z	Exact solution	$M = 2$ [14]	$M = 2$ PM	$M = 3$ [14]	$M = 3$ PM
0.2	0.0080	0.0614	0.0291	$2.55004350968591 \times 10^{-16}$	0
0.4	0.0640	0.1541	0.0885	$2.77555756156289 \times 10^{-16}$	0
0.6	0.2160	0.3339	0.1239	$3.33066907387547 \times 10^{-16}$	0
0.8	0.5120	0.6488	0.0875	$4.44089209850063 \times 10^{-16}$	0
1	1.0000	1.1468	0.0689	$6.66133814775094 \times 10^{-16}$	0

Table 8. Error table of Example 6.5 at $M = 10$ and various values of $\beta = \delta$.

z	$\beta = \delta = 0.75$	$\beta = \delta = 0.5$	$\beta = \delta = 0.67$
0.2	4.27×10^{-11}	5.87×10^{-7}	9.24×10^{-10}
0.4	8.3×10^{-9}	2.10×10^{-5}	1.40×10^{-7}
0.6	2.45×10^{-7}	3.01×10^{-4}	2.87×10^{-6}
0.8	2.73×10^{-6}	1.57×10^{-3}	2.48×10^{-5}
1	1.78×10^{-5}	5.70×10^{-3}	1.33×10^{-4}

The exact solution of (63) at $\delta = \eta = 1$ is

$$u(z) = \sum_{k=1}^{\infty} \frac{z^{\delta k}}{\Gamma(\delta k + 1)}.$$

In (63), we analyze the effects of the real number β and the fractional parameter δ by computing the approximate solution and determining the absolute errors at their various noninteger values. We observe that at various values of β and δ, the exact and approximate solutions show great resemblance with each other, see Figures 15–17. We also obtain promising results by calculating the amount of the absolute errors at $M = 10$ and various values of $\beta = \delta$, see Table 8.

Example 6.6: Consider the problem in modified GFDO of Caputo-type [27]

$$\begin{aligned} {}_{C}\mathcal{D}^{\delta,\eta}u(z) &= a\,{}_{C}\mathcal{D}^{\delta_1,\eta_1}u(z) - u(z) + F(z),\ z \in [0,1],\ 0 < \delta_1 < \delta < 1, \\ u(0) &= 0. \end{aligned} \tag{64}$$

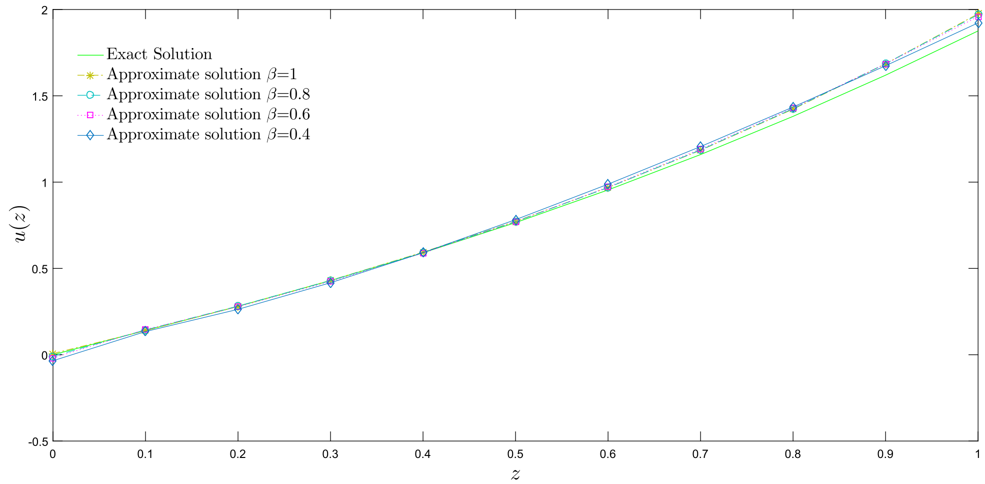

Figure 15. Exact and approximate plots of Example 6.5 at $M = 3, \delta = 0.9$, and various values of β.

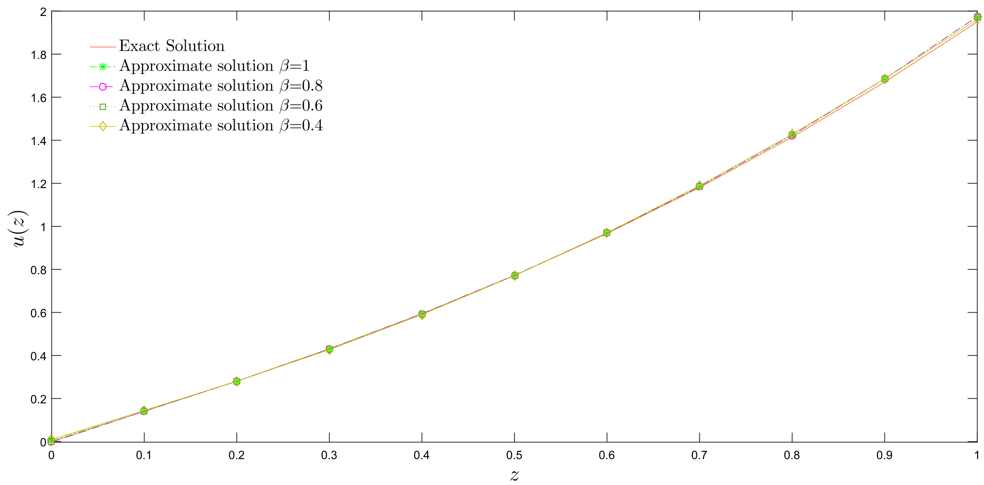

Figure 16. Exact and approximate plots of Example 6.5 at $M = 4, \delta = 0.9$, and various values of β.

The source term is

$$F(z) = \frac{5\, z^{\frac{3}{2}}}{2} + z^{\frac{5}{2}} + \frac{15\sqrt{\pi}\, z^{\frac{9}{4}}}{8\Gamma\left(\frac{13}{4}\right)}.$$

The exact solution of (64) at $\delta = 1, a = -1$, and $\delta_1 = \frac{1}{4}$ is

$$u(z) = z^2\sqrt{z}.$$

We solve (64) at various values of β and M to demonstrate the applicability of our PM. We compare the results computed by using our PM with the results obtained in [27] and [4] at various values of β and M, see Table 9. We observe that the errors computed in terms of L^∞ and L^2 by using our method are very less than those computed by using the methods presented in [27] and [4]. This highlights the efficiency of our method for this problem. Note that the symbol '$-$' means that the results for M are unavailable for the methods [27] and [4].

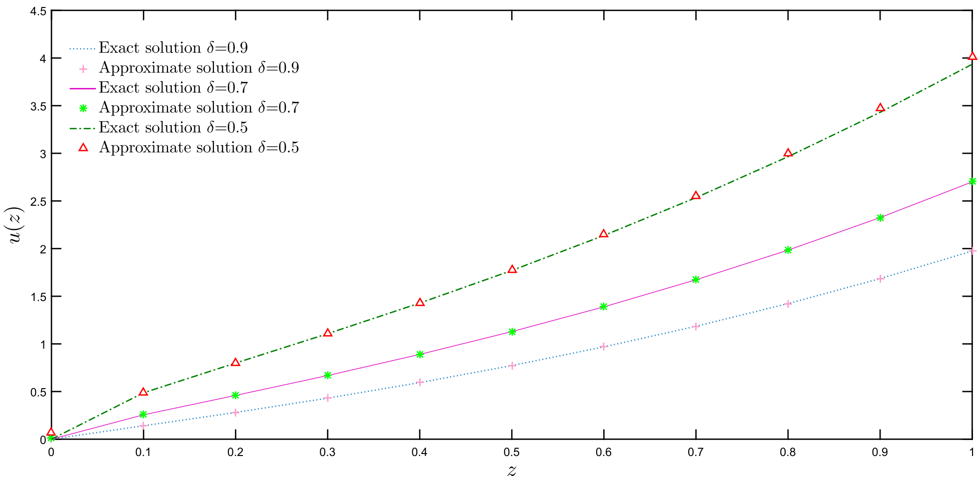

Figure 17. Exact and approximate plots of Example 6.5 at $M = 7$, $\beta = 1$, and various values of δ.

Table 9. Approximate results of Example 6.5 at various values of M and β.

		Our Method		Talaei's [27]		Al-Sharif's [4]	
M	β	L^∞	L^2	L^∞	L^2	L^∞	L^2
4	1	5.433×10^{-5}	7.81×10^{-5}	1.21×10^{-3}	5.92×10^{-4}	3.82×10^{-4}	3.81×10^{-4}
6	0.5	2.56×10^{-16}	2.22×10^{-16}	–	–	–	–
8	0.5	7.82×10^{-17}	5.55×10^{-17}	5.80×10^{-5}	2.50×10^{-5}	1.18×10^{-7}	4.06×10^{-7}
16	0.25	5.55×10^{-17}	6.03×10^{-17}	2.45×10^{-6}	9.89×10^{-7}	8.13×10^{-17}	5.36×10^{-17}
20	0.25	3.33×10^{-16}	3.52×10^{-16}	8.59×10^{-7}	3.42×10^{-7}	1.78×10^{-15}	0

7. Conclusion

In the present work, we introduced new generalized derivatives and integral operational matrices for the fractional LVF in the sense of generalized Caputo type fractional derivatives and generalized Riemann–Liouville type fractional integral operators. The operational matrices developed in the senses of Caputo and Riemann–Liouville are special cases of our newly proposed operational matrices for $\beta = \eta = 1$. Based on the generalized operational matrices, we introduced a numerical method for solving FDEs that include generalized Caputo-type fractional derivatives. The proposed method is fully dependent on the operational matrices and produces more accurate results as compared to spectral Tau and spectral collocation methods. The results computed by using our proposed method have been compared with the results obtained by using the spectral Tau method [25], function approximation theory [14], Bessel collocation method [31], Taylor matrix method [13], stochastic technique [3], and Chelyshkov collocation methods [4,27]. The comparison shows that the proposed method produced highly accurate results and only a small number of fractional LVF terms are required to obtain satisfactory results. Finally, the proposed method has the advantage of transforming FDEs into Sylvester type equations that are easy to solve by using any computational software.

Acknowledgments

The authors would like to thank both anonymous referees and the handling editor for many useful comments and suggestions, leading to a substantial improvement of the presentation of this article.

Disclosure statement

No potential conflict of interest was reported by the author(s).

ORCID

Imran Talib http://orcid.org/0000-0003-0115-4506

References

[1] B. Ahmad, M. Alghanmi, S.K. Ntouyas, and A. Alsaedi, *Fractional differential equations involving generalized derivative with Stieltjes and fractional integral boundary conditions*, Appl. Math. Lett. 84 (2018), pp. 111–117.
[2] R. Ali Khan and H. Khalil, *A new method based on legendre polynomials for solution of system of fractional order partial differential equations*, Int. J. Comput. Math. 91(12) (2014), pp. 2554–2567.
[3] J. Ali Khan, I. Qureshi, and M.M.A.Z. Raja, *Solution of fractional order system of Bagley–Torvik equation using evolutionary computational intelligence*, Math. Probl. Eng. 2011(2) (2011), pp. 675075.
[4] M.S. Al-Sharif, A.I. Ahmed, and M.S. Salim, *An integral operational matrix of fractional-order Chelyshkov functions and its applications*, Symmetry 12(11) (2020), pp. 1755.
[5] A. Atangana, *Mathematical model of survival of fractional calculus, critics and their impact: How singular is our world?*, Adv. Differ. Equ. 2021 (2021), pp. 403.
[6] A. Atangana and D. Baleanu, *New fractional derivative with non-local and non-singular kernel*, J. Therm. Sci. 20(2) (2016), pp. 763–769.
[7] D. Baleanu and R.P. Agarwal, *Fractional calculus in the sky*, Adv. Differ. Equ. 2021 (2021), pp. 117.
[8] D. Baleanu, G.-C. Wu, and S.-D. Zeng, *Chaos analysis and asymptotic stability of generalized Caputo fractional differential equations*, Chaos Solitons Fractals 102 (2017), pp. 99–105.
[9] M. Benjemaa, *Taylor's formula involving generalized fractional derivatives*, Appl. Math. Comput. 335 (2018), pp. 182–195.
[10] E.H. Doha, A.H. Bhrawy, and S.S. Ezz-Eldien, *A new Jacobi operational matrix: An application for solving fractional differential equations*, Appl. Math. Model. 36(10) (2012), pp. 4931–4943.
[11] A.N. Fall, S.N. Ndiaye, and N. Sene, *Black-Scholes option pricing equations described by the Caputo generalized fractional derivative*, Chaos Solitons Fractals 125 (2019), pp. 108–118.
[12] S. Gaboury, R. Tremblay, and B.-J. Fugère, *Some relations involving a generalized fractional derivative operator*, J. Inequal. Appl. 2013 (2013), pp. 167.
[13] M. Gülsu, Y. Öztürk, and A. Anapali, *Numerical solution the fractional Bagley–Torvik equation arising in fluid mechanics*, Int. J. Comput. Math. 94(1) (2017), pp. 173–184.
[14] W. Han, Y.-M. Chen, D.-Y. Liu, X.-L. Li, and D. Boutat, *Numerical solution for a class of multi-order fractional differential equations with error correction and convergence analysis*, Adv. Differ. Equ. 2018 (2018), pp. 1–22.
[15] R. Hilfer (ed.), *Applications of Fractional Calculus in Physics*, World Scientific Publishing Co., Inc., River Edge, NJ, 2000.
[16] F. Jarad, T. Abdeljawad, and D. Baleanu, *On the generalized fractional derivatives and their Caputo modification*, J. Nonlinear Sci. Appl. 10(5) (2017), pp. 2607–2619.
[17] U.N. Katugampola, *New approach to a generalized fractional integral*, Appl. Math. Comput. 218(3) (2011), pp. 860–865.
[18] U.N. Katugampola, *A new approach to generalized fractional derivatives*, Bull. Math. Anal. Appl.6(4) (2014), pp. 1–15.
[19] S. Kazem, S. Abbasbandy, and S. Kumar, *Fractional-order Legendre functions for solving fractional-order differential equations*, Appl. Math. Model. 37(7) (2013), pp. 5498–5510.
[20] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies Vol. 204, Elsevier Science B.V., Amsterdam, 2006.
[21] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, A Wiley-Interscience Publication, John Wiley & Sons Inc., New York, 1993.
[22] Z. Odibat and D. Baleanu, *Numerical simulation of initial value problems with generalized Caputo-type fractional derivatives*, Appl. Numer. Math. 156 (2020), pp. 94–105.
[23] I. Podlubny, *An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, in *Fractional Differential Equations*, Mathematics in Science and Engineering Vol. 198, Academic Press, Inc., San Diego, 1999.
[24] S.Z. Rida and A.M. Yousef, *On the fractional order Rodrigues formula for the Legendre polynomials* $P_{\frac{1}{n}}(x)$, Adv. Appl. Math. Sci. 10(5) (2011), pp. 509–517.
[25] A. Saadatmandi and M. Dehghan, *A new operational matrix for solving fractional-order differential equations*, Comput. Math. Appl. 59(3) (2010), pp. 1326–1336.
[26] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach Science Publishers, Yverdon, 1993, Theory and applications, edited and with a foreword by S. M. Nikol skĭ;, translated from the 1987 Russian original, revised by the authors.

[27] Y. Talaei and M. Asgari, *An operational matrix based on Chelyshkov polynomials for solving multi-order fractional differential equationsnumerical solution the fractional bagley–torvik equation arising in fluid mechanics*, Neural. Comput. Appl. 30 (2018), pp. 1369–1376.

[28] I. Talib, F. Jarad, M.U. Mirza, A. Nawaz, and M.B. Riaz, *A generalized operational matrix of mixed partial derivative terms with applications to multi-order fractional partial differential equations*, Alex. Eng. J. 61(1) (2022), pp. 135–145.

[29] I. Talib, M. Nur Alam, D. Baleanu, D. Zaidi, and A. Marriyam, *A new integral operational matrix with applications to multi-order fractional differential equations*, AIMS Math. 6(8) (2021), pp. 8742–8771.

[30] I. Talib, A. Nur, D. Baleanu, and Z. Danish, *A decomposition algorithm coupled with operational matrices approach with applications to fractional differential equations*, J. Therm. Sci. 25(S2) (2021), pp. 449–455.

[31] Ş. Yüzbaşı, *Numerical solution of the Bagley-Torvik equation by the Bessel collocation method*, Math. Methods Appl. Sci. 36(3) (2013), pp. 300–312.

[32] S. Zeng, D. Baleanu, Y. Bai, and G. Wu, *Fractional differential equations of Caputo-Katugampola type and numerical solutions*, Appl. Math. Comput. 315 (2017), pp. 549–554.