



Oscillation criteria for second-order dynamic equations on time scales



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ABSTRACT

Erbe's and Hassan's contributions regarding oscillation criteria are interesting in the development of oscillation theory of dynamic equations on time scales. The objective of this paper is to amend these results.

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1. Introduction

The increasing interest in oscillatory properties of solutions to dynamic equations on time scales is motivated by their applications in the engineering and natural sciences. We refer the reader to [1–19] and the references cited therein. In [10] and [12], the authors studied the oscillatory behavior of second-order dynamic equations

$$(r(t)x^\Delta(t))^\Delta + p(t)f(x(\tau(t))) = 0 \quad (1.1)$$

and

$$(r(t)(x^\Delta(t))^\gamma)^\Delta + p(t)x^\gamma(t) = 0, \quad (1.2)$$

respectively, where γ is the quotient of odd positive integers and

(A₁) r and p are positive real-valued rd-continuous functions defined on $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$;

(A₂) the delay function $\tau \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ satisfies $\tau(t) \leq t$ and $\lim_{t \rightarrow \infty} \tau(t) = \infty$;

(A₃) $f \in C(\mathbb{R}, \mathbb{R})$ such that $yf(y) > 0$, $f(y)/y \geq K > 0$ for $y \neq 0$, where K is a constant.

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A time scale \mathbb{T} is an arbitrary nonempty closed subset of the real numbers \mathbb{R} . On any time scale we define the forward and backward jump operators by $\sigma(t) := \inf\{s \in \mathbb{T} | s > t\}$ and $\rho(t) := \sup\{s \in \mathbb{T} | s < t\}$, where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$, \emptyset denotes the empty set. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, right-dense if $\sigma(t) = t$ and $t < \sup \mathbb{T}$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. Points that are right-scattered and left-scattered at the same time are called isolated. Regarding the time scales that consist of only isolated points; see, for example, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$, $\mathbb{T} = q^{\mathbb{N}}$, and $\mathbb{T} = 2^{\mathbb{N}}$, etc. The graininess function $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \rightarrow \mathbb{R}$ the notation $f^\sigma(t) := f(\sigma(t))$. Some other concepts related to the notion of time scales; see Bohner and Peterson [6,7].

We assume that solutions of (1.1) (or (1.2)) exist for any $t \in [t_0, \infty)_{\mathbb{T}}$. A solution x of (1.1) (or (1.2)) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, we call it nonoscillatory. Eq. (1.1) (or (1.2)) is said to be oscillatory if all its solutions oscillate.

In order to derive oscillation results for (1.1), Erbe et al. [10] utilized the class of functions as follows: $H \in \mathfrak{H}$ if H is defined for $t_0 \leq s \leq \sigma(t)$, $t, s \in [t_0, \infty)_{\mathbb{T}}$, $H(t, s) \geq 0$, and satisfies $H(\sigma(t), t) = 0$, $H^{\Delta_s}(t, s) \leq 0$ for $t > s \geq t_0$, and for each fixed t , $H^{\Delta_s}(t, s)$ is delta integrable with respect to s . For completeness, we present one of the results in [10] as below.

Theorem 1.1 (See [10, Theorem 1]). Assume that (A_1) – (A_3) are satisfied and let

$$r^{\Delta}(t) \geq 0, \quad \int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \infty, \quad \text{and} \quad \int_{t_0}^{\infty} p(t)\tau(t)\Delta t = \infty. \tag{1.3}$$

Suppose further that there exist a function η and a positive, differentiable function δ such that for some $H \in \mathfrak{H}$ and for sufficiently large t_1 ,

$$\limsup_{t \rightarrow \infty} \frac{1}{H(\sigma(t), t_1)} \int_{t_1}^t H(\sigma(t), \sigma(s))\delta^\sigma(s) [\psi(s) - \phi(t, s)] \Delta s = \infty, \tag{1.4}$$

where

$$\begin{aligned} \phi(t, s) &:= \frac{1}{4} \left(\frac{\delta(s)}{\delta^\sigma(s)} \right)^2 \frac{r(s)A^2(t, s)}{C(s)}, \quad C(t) := \frac{t}{\sigma(t)}, \\ \psi(s) &:= \frac{Kp(s)\tau(s)}{\sigma(s)} - (\eta(s)r(s))^\Delta + \frac{sr(s)\eta^2(s)}{\sigma(s)}, \end{aligned}$$

and

$$A(t, s) := \frac{\delta^\sigma(s)C_1(s)}{\delta(s)} + \frac{H^{\Delta_s}(\sigma(t), s)}{H(\sigma(t), \sigma(s))}, \quad C_1(s) := \frac{\delta^\Delta(s)}{\delta^\sigma(s)} + 2\frac{s\eta(s)}{\sigma(s)}.$$

Then (1.1) is oscillatory.

To prove Theorem 1.1, the authors defined a generalized Riccati substitution

$$w(t) := \delta(t) \left[\frac{r(t)x^\Delta(t)}{x(t)} + r(t)\eta(t) \right], \tag{1.5}$$

and then derived the following formula; see [10, (2.15)]

$$\begin{aligned} \int_{t_2}^t H(\sigma(t), \sigma(s))\delta^\sigma(s)\psi(s)\Delta s &\leq H(\sigma(t), t_2)w(t_2) - \int_{t_2}^t H(\sigma(t), \sigma(s))\frac{C(s)\delta^\sigma(s)}{r(s)\delta^2(s)}w^2(s)\Delta s \\ &\quad + \int_{t_2}^t H(\sigma(t), \sigma(s))A(t, s)w(s)\Delta s. \end{aligned} \tag{1.6}$$

Then by this inequality and using the method of completing the square, they obtained

$$\int_{t_2}^t H(\sigma(t), \sigma(s))\delta^\sigma(s)\psi(s)\Delta s \leq H(\sigma(t), t_2)w(t_2) + \int_{t_2}^t H(\sigma(t), \sigma(s))\frac{r(s)\delta^2(s)A^2(t, s)}{4C(s)\delta^\sigma(s)}\Delta s. \tag{1.7}$$

Note that when the time scale \mathbb{T} considered only contains isolated points, e.g., $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$, and $\mathbb{T} = \{t : t = q^k, k \in \mathbb{N}_0, q > 1\}$, etc., completing the square cannot be applied in inequality (1.6) to provide (1.7) since $H(\sigma(t), \sigma(\rho(t))) = H(\sigma(t), t) = 0$.

To present oscillation theorems for (1.2), Hassan [12] employed the class of functions as follows: $H \in \mathfrak{H}$ if $H : [t_0, \infty)_{\mathbb{T}} \times [t_0, \infty)_{\mathbb{T}} \rightarrow \mathbb{R}$ and satisfies $H(t, t) = 0$, $t \geq t_0$, $H(t, s) > 0$, $t > s \geq t_0$. In the following, we give one of the results presented in [12] for the convenience of the reader.

Theorem 1.2 (See [12, Theorem 2.2]). Assume that (A_1) is satisfied and let

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{\frac{1}{\gamma}}(t)} = \infty. \quad (1.8)$$

Let $H \in \mathfrak{R}$ be such that H has a nonpositive rd-continuous Δ -partial derivative $H^{\Delta s}(t, s)$ with respect to the second variable and satisfies

$$H^{\Delta s}(\sigma(t), s) + H(\sigma(t), \sigma(s)) \frac{\delta^{\Delta}(s)}{\delta(s)} = -\frac{h(t, s)}{\delta(s)} (H(\sigma(t), \sigma(s)))^{\frac{\gamma}{\gamma+1}},$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{H(\sigma(t), t_0)} \int_{t_0}^{\sigma(t)} K(t, s) \Delta s = \infty, \quad (1.9)$$

where δ is a positive Δ -differentiable function and

$$K(t, s) := H(\sigma(t), \sigma(s)) \alpha^{\gamma}(s) \delta^{\sigma}(s) p(s) - \frac{r(s)(h_-(t, s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\delta^{\sigma}(s))^{\gamma}},$$

$$\beta(t) := \begin{cases} \alpha(t), & 0 < \gamma \leq 1, \\ \alpha^{\gamma}(t), & \gamma > 1, \end{cases}$$

$$\alpha(t) := \frac{R(t)}{R(t) + \mu(t)}, \quad R(t) := r^{\frac{1}{\gamma}}(t) \int_{t_0}^t \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}, \quad h_-(t, s) := \max\{0, -h(t, s)\}.$$

Then (1.2) is oscillatory.

In the next section, we show that Theorem 1.2 is invalid for Eq. (1.2) on an isolated time scale. Hence the purpose of this paper is to amend Theorems 1.1 and 1.2.

2. Main results

All functional inequalities considered in this section are assumed to hold eventually, that is, they are satisfied for all t large enough. We begin with the following lemma which can be found in [12].

Lemma 2.1 (See [12, Lemma 2.1]). Assume that (A_1) and (1.8) are satisfied, and let x be a positive solution of (1.2) on $[t_0, \infty)_{\mathbb{T}}$. Then

$$(r(x^{\Delta})^{\gamma})^{\Delta}(t) < 0 \quad \text{and} \quad x^{\Delta}(t) > 0$$

for $t \in [t_0, \infty)_{\mathbb{T}}$.

Theorem 2.2. Assume that (A_1) – (A_3) and (1.3) are satisfied, and let

$$\sigma(t) > t \quad \text{and} \quad \rho(t) < t \quad \text{for all } t \in [t_0, \infty)_{\mathbb{T}}. \quad (2.1)$$

Assume further that there exist a nonnegative function η and a positive, differentiable function δ such that for some $H \in \mathfrak{S}$ and for sufficiently large t_1 ,

$$\limsup_{t \rightarrow \infty} \frac{1}{H(\sigma(t), t_1)} \sum_{s=t_1}^{\rho(\sigma(t))} \mu(s) H(\sigma(t), \sigma(s)) \left[\delta^{\sigma}(s) \psi(s) - \frac{r(s) \delta^2(s) A^2(t, s)}{4C(s) \delta^{\sigma}(s)} \right] = \infty, \quad (2.2)$$

where ψ , A , and C are as in Theorem 1.1. Then (1.1) is oscillatory.

Proof. Assume that (1.1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, suppose that it is an eventually positive solution. Define function w by (1.5). Proceeding as in the proof of [10, Theorem 1], we obtain

$$\int_{t_2}^t H(\sigma(t), \sigma(s)) \delta^{\sigma}(s) \psi(s) \Delta s \leq H(\sigma(t), t_2) w(t_2) - \int_{t_2}^t H(\sigma(t), \sigma(s)) \frac{C(s) \delta^{\sigma}(s)}{r(s) \delta^2(s)} w^2(s) \Delta s$$

$$+ \int_{t_2}^t \left[H(\sigma(t), \sigma(s)) \frac{\delta^{\sigma}(s) C_1(s)}{\delta(s)} + H^{\Delta s}(\sigma(t), s) \right] w(s) \Delta s. \quad (2.3)$$

For the case where \mathbb{T} only contains isolated points, we have

$$\begin{aligned} & - \int_{t_2}^t H(\sigma(t), \sigma(s)) \frac{C(s)\delta^\sigma(s)}{r(s)\delta^2(s)} w^2(s) \Delta s + \int_{t_2}^t \left[H(\sigma(t), \sigma(s)) \frac{\delta^\sigma(s)C_1(s)}{\delta(s)} + H^{\Delta s}(\sigma(t), s) \right] w(s) \Delta s \\ &= - \int_{t_2}^{\rho(t)} H(\sigma(t), \sigma(s)) \frac{C(s)\delta^\sigma(s)}{r(s)\delta^2(s)} w^2(s) \Delta s + \int_{t_2}^{\rho(t)} H(\sigma(t), \sigma(s)) A(t, s) w(s) \Delta s \\ & \quad - \int_{\rho(t)}^t H(\sigma(t), \sigma(s)) \frac{C(s)\delta^\sigma(s)}{r(s)\delta^2(s)} w^2(s) \Delta s + \int_{\rho(t)}^t \left[H(\sigma(t), \sigma(s)) \frac{\delta^\sigma(s)C_1(s)}{\delta(s)} + H^{\Delta s}(\sigma(t), s) \right] w(s) \Delta s \\ &= - \int_{t_2}^{\rho(t)} H(\sigma(t), \sigma(s)) \frac{C(s)\delta^\sigma(s)}{r(s)\delta^2(s)} w^2(s) \Delta s + \int_{t_2}^{\rho(t)} H(\sigma(t), \sigma(s)) A(t, s) w(s) \Delta s \\ & \quad - \mu(\rho(t)) H(\sigma(t), \sigma(\rho(t))) \frac{C(\rho(t))\delta^\sigma(\rho(t))}{r(\rho(t))\delta^2(\rho(t))} w^2(\rho(t)) \\ & \quad + \mu(\rho(t)) H(\sigma(t), \sigma(\rho(t))) \frac{\delta^\sigma(\rho(t))C_1(\rho(t))}{\delta(\rho(t))} w(\rho(t)) + \mu(\rho(t)) H^{\Delta s}(\sigma(t), \rho(t)) w(\rho(t)) \\ &= - \int_{t_2}^{\rho(t)} H(\sigma(t), \sigma(s)) \frac{C(s)\delta^\sigma(s)}{r(s)\delta^2(s)} w^2(s) \Delta s + \int_{t_2}^{\rho(t)} H(\sigma(t), \sigma(s)) A(t, s) w(s) \Delta s \\ & \quad + \mu(\rho(t)) H^{\Delta s}(\sigma(t), \rho(t)) w(\rho(t)). \end{aligned}$$

On the basis of $a(t) \geq 0$, one has $\mu(\rho(t)) H^{\Delta s}(\sigma(t), \rho(t)) w(\rho(t)) \leq 0$, and so

$$\begin{aligned} & - \int_{t_2}^t H(\sigma(t), \sigma(s)) \frac{C(s)\delta^\sigma(s)}{r(s)\delta^2(s)} w^2(s) \Delta s + \int_{t_2}^t \left[H(\sigma(t), \sigma(s)) \frac{\delta^\sigma(s)C_1(s)}{\delta(s)} + H^{\Delta s}(\sigma(t), s) \right] w(s) \Delta s \\ & \leq - \int_{t_2}^{\rho(t)} H(\sigma(t), \sigma(s)) \frac{C(s)\delta^\sigma(s)}{r(s)\delta^2(s)} w^2(s) \Delta s + \int_{t_2}^{\rho(t)} H(\sigma(t), \sigma(s)) A(t, s) w(s) \Delta s \\ & \leq \sum_{s=t_2}^{\rho(\rho(t))} \mu(s) H(\sigma(t), \sigma(s)) \frac{r(s)\delta^2(s)A^2(t, s)}{4C(s)\delta^\sigma(s)}. \end{aligned}$$

Then by $H(\sigma(t), t) = 0$ and (2.3), we get

$$\begin{aligned} & \int_{t_2}^t H(\sigma(t), \sigma(s)) \delta^\sigma(s) \psi(s) \Delta s - \sum_{s=t_2}^{\rho(\rho(t))} \mu(s) H(\sigma(t), \sigma(s)) \frac{r(s)\delta^2(s)A^2(t, s)}{4C(s)\delta^\sigma(s)} \\ &= \sum_{s=t_2}^{\rho(t)} \mu(s) H(\sigma(t), \sigma(s)) \delta^\sigma(s) \psi(s) - \sum_{s=t_2}^{\rho(\rho(t))} \mu(s) H(\sigma(t), \sigma(s)) \frac{r(s)\delta^2(s)A^2(t, s)}{4C(s)\delta^\sigma(s)} \\ &= \sum_{s=t_2}^{\rho(\rho(t))} \mu(s) H(\sigma(t), \sigma(s)) \left[\delta^\sigma(s) \psi(s) - \frac{r(s)\delta^2(s)A^2(t, s)}{4C(s)\delta^\sigma(s)} \right] \\ & \leq H(\sigma(t), t_2) w(t_2), \end{aligned}$$

which yields

$$\frac{1}{H(\sigma(t), t_2)} \sum_{s=t_2}^{\rho(\rho(t))} \mu(s) H(\sigma(t), \sigma(s)) \left[\delta^\sigma(s) \psi(s) - \frac{r(s)\delta^2(s)A^2(t, s)}{4C(s)\delta^\sigma(s)} \right] \leq w(t_2).$$

This contradicts (2.2). The proof is complete. \square

Remark 2.3. Theorem 2.2 amends Theorem 1.1 and can be applied to the cases where $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$, $\mathbb{T} = \{t : t = q^k, k \in \mathbb{N}_0, q > 1\}$, and other specific discrete time scales of interest. As a matter of fact, we replace $\int_{t_1}^t$ in (1.4) with $\int_{t_1}^{\rho(t)}$ in (2.2) assuming that (2.1) is satisfied.

Example 2.4. Consider a second-order delay difference equation

$$x^{\Delta\Delta}(t) + \frac{t+1}{t-1} x(t-1) = 0, \tag{2.4}$$

where $t \in [2, \infty)_{\mathbb{Z}}$. Let $K = 1$, $\delta(t) = 1$, $\eta(t) = 0$, and $H(t, s) = (t - s)(t - s - 1)$. Then

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{H(\sigma(t), t_1)} \sum_{s=t_1}^{\rho(\sigma(t))} \mu(s) H(\sigma(t), \sigma(s)) \left[\delta^\sigma(s) \psi(s) - \frac{r(s) \delta^2(s) A^2(t, s)}{4C(s) \delta^\sigma(s)} \right] \\ &= \limsup_{t \rightarrow \infty} \frac{1}{(t - t_1)(t + 1 - t_1)} \sum_{s=t_1}^{t-2} \left((t - s)(t - s - 1) - \frac{t - s}{t - s - 1} \frac{s + 1}{s} \right) \\ &\geq \limsup_{t \rightarrow \infty} \frac{1}{(t - t_1)(t + 1 - t_1)} \sum_{s=t_1}^{t-2} (t^2 - 2ts - t + s^2 + s - 4) = \infty. \end{aligned}$$

Therefore, we conclude that Eq. (2.4) is oscillatory by Theorem 2.2.

Theorem 2.5. Assume that (A_1) and (1.8) are satisfied. Let $H \in \mathfrak{H}$ be such that H has a nonpositive rd-continuous Δ -partial derivative $H^{\Delta_s}(t, s)$ with respect to the second variable and satisfies

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{H(\sigma(t), t_0)} \int_{t_0}^t \left[H(\sigma(t), \sigma(s)) \alpha^\gamma(s) \delta^\sigma(s) p(s) \right. \\ & \quad \left. - \frac{r(s) \left[\left(H(\sigma(t), \sigma(s)) \frac{\delta^\Delta(s)}{\delta(s)} + H^{\Delta_s}(\sigma(t), s) \right)_+ \right]^{\gamma+1} \delta^{\gamma+1}(s)}{(\gamma + 1)^{\gamma+1} (\beta(s) \delta^\sigma(s) H(\sigma(t), \sigma(s)))^\gamma} \right] \Delta s = \infty, \end{aligned} \tag{2.5}$$

where α, β , and δ are as in Theorem 1.2 and

$$\left(H(\sigma(t), \sigma(s)) \frac{\delta^\Delta(s)}{\delta(s)} + H^{\Delta_s}(\sigma(t), s) \right)_+ := \max \left\{ 0, H(\sigma(t), \sigma(s)) \frac{\delta^\Delta(s)}{\delta(s)} + H^{\Delta_s}(\sigma(t), s) \right\}.$$

Then (1.2) is oscillatory.

Proof. Assume (1.2) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there is a $T \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$ on $[T, \infty)_{\mathbb{T}}$. Consider the generalized Riccati substitution

$$w(t) := \delta(t) r(t) \left(\frac{x^\Delta(t)}{x(t)} \right)^\gamma.$$

Then we have by Lemma 2.1 that $w(t) > 0$. From the proof of [12, Theorem 2.1], we have, for $\gamma > 0$,

$$w^\Delta(t) \leq -\alpha^\gamma(t) \delta^\sigma(t) p(t) + \frac{\delta^\Delta(t)}{\delta(t)} w(t) - \frac{\gamma \beta(t) \delta^\sigma(t)}{\delta^\lambda(t) r^{\lambda-1}(t)} w^\lambda(t),$$

where $\lambda := (\gamma + 1)/\gamma$; see [12, (2.8)]. That is,

$$\alpha^\gamma(t) \delta^\sigma(t) p(t) \leq -w^\Delta(t) + \frac{\delta^\Delta(t)}{\delta(t)} w(t) - \frac{\gamma \beta(t) \delta^\sigma(t)}{\delta^\lambda(t) r^{\lambda-1}(t)} w^\lambda(t). \tag{2.6}$$

Multiplying both sides of (2.6), with t replaced by s , by $H(\sigma(t), \sigma(s))$, integrating with respect to s from T to $\sigma(t)$, we get

$$\begin{aligned} \int_T^{\sigma(t)} H(\sigma(t), \sigma(s)) \alpha^\gamma(s) \delta^\sigma(s) p(s) \Delta s &\leq - \int_T^{\sigma(t)} H(\sigma(t), \sigma(s)) w^\Delta(s) \Delta s + \int_T^{\sigma(t)} H(\sigma(t), \sigma(s)) \frac{\delta^\Delta(s)}{\delta(s)} w(s) \Delta s \\ &\quad - \int_T^{\sigma(t)} H(\sigma(t), \sigma(s)) \frac{\gamma \beta(s) \delta^\sigma(s)}{\delta^\lambda(s) r^{\lambda-1}(s)} w^\lambda(s) \Delta s. \end{aligned}$$

Integrating by parts, we have

$$\begin{aligned} \int_T^{\sigma(t)} H(\sigma(t), \sigma(s)) \alpha^\gamma(s) \delta^\sigma(s) p(s) \Delta s &\leq H(\sigma(t), T) w(T) + \int_T^{\sigma(t)} H^{\Delta_s}(\sigma(t), s) w(s) \Delta s \\ &\quad + \int_T^{\sigma(t)} H(\sigma(t), \sigma(s)) \frac{\delta^\Delta(s)}{\delta(s)} w(s) \Delta s \\ &\quad - \int_T^{\sigma(t)} H(\sigma(t), \sigma(s)) \frac{\gamma \beta(s) \delta^\sigma(s)}{\delta^\lambda(s) r^{\lambda-1}(s)} w^\lambda(s) \Delta s. \end{aligned} \tag{2.7}$$

Now, we have by $\mu(t)H^{\Delta_s}(\sigma(t), t)w(t) \leq 0$ that

$$\begin{aligned} & \int_T^{\sigma(t)} \left[H(\sigma(t), \sigma(s)) \frac{\delta^{\Delta}(s)}{\delta(s)} + H^{\Delta_s}(\sigma(t), s) \right] w(s) \Delta s - \int_T^{\sigma(t)} H(\sigma(t), \sigma(s)) \frac{\gamma \beta(s) \delta^{\sigma}(s)}{\delta^{\lambda}(s) r^{\lambda-1}(s)} w^{\lambda}(s) \Delta s \\ &= \int_T^t \left[H(\sigma(t), \sigma(s)) \frac{\delta^{\Delta}(s)}{\delta(s)} + H^{\Delta_s}(\sigma(t), s) \right] w(s) \Delta s - \int_T^t H(\sigma(t), \sigma(s)) \frac{\gamma \beta(s) \delta^{\sigma}(s)}{\delta^{\lambda}(s) r^{\lambda-1}(s)} w^{\lambda}(s) \Delta s \\ &+ \int_t^{\sigma(t)} \left[H(\sigma(t), \sigma(s)) \frac{\delta^{\Delta}(s)}{\delta(s)} + H^{\Delta_s}(\sigma(t), s) \right] w(s) \Delta s - \int_t^{\sigma(t)} H(\sigma(t), \sigma(s)) \frac{\gamma \beta(s) \delta^{\sigma}(s)}{\delta^{\lambda}(s) r^{\lambda-1}(s)} w^{\lambda}(s) \Delta s \\ &= \int_T^t \left[H(\sigma(t), \sigma(s)) \frac{\delta^{\Delta}(s)}{\delta(s)} + H^{\Delta_s}(\sigma(t), s) \right] w(s) \Delta s - \int_T^t H(\sigma(t), \sigma(s)) \frac{\gamma \beta(s) \delta^{\sigma}(s)}{\delta^{\lambda}(s) r^{\lambda-1}(s)} w^{\lambda}(s) \Delta s \\ &+ \mu(t) \left[H(\sigma(t), \sigma(t)) \frac{\delta^{\Delta}(t)}{\delta(t)} + H^{\Delta_s}(\sigma(t), t) \right] w(t) - \mu(t) H(\sigma(t), \sigma(t)) \frac{\gamma \beta(t) \delta^{\sigma}(t)}{\delta^{\lambda}(t) r^{\lambda-1}(t)} w^{\lambda}(t) \\ &\leq \int_T^t \left[H(\sigma(t), \sigma(s)) \frac{\delta^{\Delta}(s)}{\delta(s)} + H^{\Delta_s}(\sigma(t), s) \right]_+ w(s) \Delta s - \int_T^t H(\sigma(t), \sigma(s)) \frac{\gamma \beta(s) \delta^{\sigma}(s)}{\delta^{\lambda}(s) r^{\lambda-1}(s)} w^{\lambda}(s) \Delta s. \end{aligned}$$

Using inequality (see [13,19])

$$Bw - Aw^{1+\frac{1}{\gamma}} \leq \frac{\gamma^{\gamma}}{(1+\gamma)^{1+\gamma}} \frac{B^{1+\gamma}}{A^{\gamma}}, \quad A > 0, \tag{2.8}$$

we obtain

$$\begin{aligned} & \int_T^{\sigma(t)} \left[H(\sigma(t), \sigma(s)) \frac{\delta^{\Delta}(s)}{\delta(s)} + H^{\Delta_s}(\sigma(t), s) \right] w(s) \Delta s - \int_T^{\sigma(t)} H(\sigma(t), \sigma(s)) \frac{\gamma \beta(s) \delta^{\sigma}(s)}{\delta^{\lambda}(s) r^{\lambda-1}(s)} w^{\lambda}(s) \Delta s \\ &\leq \int_T^t \frac{r(s) \left[\left(H(\sigma(t), \sigma(s)) \frac{\delta^{\Delta}(s)}{\delta(s)} + H^{\Delta_s}(\sigma(t), s) \right)_+ \right]^{\gamma+1} \delta^{\gamma+1}(s)}{(\gamma+1)^{\gamma+1} (\beta(s) \delta^{\sigma}(s) H(\sigma(t), \sigma(s)))^{\gamma}} \Delta s. \end{aligned}$$

Thus, we have by (2.7) and $H(\sigma(t), \sigma(t)) = 0$ that

$$\begin{aligned} & \int_T^t H(\sigma(t), \sigma(s)) \alpha^{\gamma}(s) \delta^{\sigma}(s) p(s) \Delta s - \int_T^t \frac{r(s) \left[\left(H(\sigma(t), \sigma(s)) \frac{\delta^{\Delta}(s)}{\delta(s)} + H^{\Delta_s}(\sigma(t), s) \right)_+ \right]^{\gamma+1} \delta^{\gamma+1}(s)}{(\gamma+1)^{\gamma+1} (\beta(s) \delta^{\sigma}(s) H(\sigma(t), \sigma(s)))^{\gamma}} \Delta s \\ &\leq H(\sigma(t), T)w(T) \leq H(\sigma(t), t_0)w(T), \end{aligned}$$

and, correspondingly,

$$\begin{aligned} & \int_{t_0}^t \left[H(\sigma(t), \sigma(s)) \alpha^{\gamma}(s) \delta^{\sigma}(s) p(s) - \frac{r(s) \left[\left(H(\sigma(t), \sigma(s)) \frac{\delta^{\Delta}(s)}{\delta(s)} + H^{\Delta_s}(\sigma(t), s) \right)_+ \right]^{\gamma+1} \delta^{\gamma+1}(s)}{(\gamma+1)^{\gamma+1} (\beta(s) \delta^{\sigma}(s) H(\sigma(t), \sigma(s)))^{\gamma}} \right] \Delta s \\ &\leq H(\sigma(t), t_0) \left[w(T) + \int_{t_0}^T \alpha^{\gamma}(s) \delta^{\sigma}(s) p(s) \Delta s \right]. \end{aligned}$$

This yields

$$\begin{aligned} & \frac{1}{H(\sigma(t), t_0)} \int_{t_0}^t \left[H(\sigma(t), \sigma(s)) \alpha^{\gamma}(s) \delta^{\sigma}(s) p(s) - \frac{r(s) \left[\left(H(\sigma(t), \sigma(s)) \frac{\delta^{\Delta}(s)}{\delta(s)} + H^{\Delta_s}(\sigma(t), s) \right)_+ \right]^{\gamma+1} \delta^{\gamma+1}(s)}{(\gamma+1)^{\gamma+1} (\beta(s) \delta^{\sigma}(s) H(\sigma(t), \sigma(s)))^{\gamma}} \right] \Delta s \\ &\leq w(T) + \int_{t_0}^T \alpha^{\gamma}(s) \delta^{\sigma}(s) p(s) \Delta s < \infty, \end{aligned}$$

which contradicts (2.5). The proof is complete. \square

Remark 2.6. Assume \mathbb{T} is an isolated time scale. From (2.7), one cannot derive a contradiction to (1.9) by using inequality (2.8), since $H(\sigma(t), \sigma(t)) = 0$. Therefore, Theorem 2.5 amends Theorem 1.2. In fact, we replace $\int_T^{\sigma(t)}$ in (1.9) with $\int_{t_0}^t$ in (2.5). On the basis of Theorem 2.5, one can easily revise related results in the papers [11,14,15].

By virtue of Theorem 2.5, we obtain the following result.

Corollary 2.7. Assume that (A_1) , (1.8) and (2.1) are satisfied. Let $H \in \mathfrak{N}$ be such that H has a nonpositive rd-continuous Δ -partial derivative $H^{\Delta s}(t, s)$ with respect to the second variable and satisfies

$$\limsup_{t \rightarrow \infty} \frac{1}{H(\sigma(t), t_0)} \sum_{s=t_0}^{\rho(t)} \mu(s) \left[H(\sigma(t), \sigma(s)) \alpha^\gamma(s) \delta^\sigma(s) p(s) - \frac{r(s) \left[\left(H(\sigma(t), \sigma(s)) \frac{\delta^\Delta(s)}{\delta(s)} + H^{\Delta s}(\sigma(t), s) \right)_+^{\gamma+1} \delta^{\gamma+1}(s) \right]}{(\gamma+1)^{\gamma+1} (\beta(s) \delta^\sigma(s) H(\sigma(t), \sigma(s)))^\gamma} \right] = \infty,$$

where α , β , and δ are as in Theorem 1.2. Then (1.2) is oscillatory.

Example 2.8. Consider a second-order delay q -difference equation

$$x^{\Delta\Delta}(t) + \frac{1}{t}x(t) = 0, \quad t \in [1, \infty)_{\mathbb{T}}, \quad (2.9)$$

where $\mathbb{T} := \overline{q\mathbb{Z}} = \{q^k : k \in \mathbb{Z}, q > 1\} \cup \{0\}$. Let $\delta(t) = 1$ and $H(t, s) = (t-s)^2$. Then a straightforward computation shows that all assumptions of Theorem 2.5 are satisfied. Therefore, Eq. (2.9) is oscillatory.

Remark 2.9. One can prove that Theorem 2.5 is valid for the second-order half-linear dynamic equation of the form

$$(r(t)\Phi(x^\Delta(t)))^\Delta + p(t)\Phi(x(t)) = 0,$$

where $\Phi(u) := |u|^\gamma \operatorname{sgn} u$ and $\gamma > 0$ is a constant. The details are left to the reader.

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