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QUALITATIVE ANALYSIS OF INTEGRO-DIFFERENTIAL EQUATIONS WITH VARIABLE RETARDATION

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Abstract. The paper is concerned with a class of nonlinear time-varying retarded integro-differential equations (RIDEs). By the Lyapunov–Krasovskiĭ functional method, two new results with weaker conditions related to uniform stability (US), uniform asymptotic stability (UAS), integrability, boundedness, and boundedness at infinity of solutions of the RIDEs are given. For illustrative purposes, two examples are provided. The study of the results of this paper shows that the given theorems are not only applicable to time-varying linear RIDEs, but also applicable to time-varying nonlinear RIDEs.

1. **Introduction.**

RIDEs appear as mathematical models of numerous scientific and engineering problems. Essentially, many applications related to science and engineering are expressed by RIDEs (see [2, 3, 10–12, 23, 29, 40, 58] and the bibliographies therein). For instance, population growth models, mathematical models of biological species living together, mathematical models of heat transfer and radiation, standard closed electric RLC circuits, etc., can be described by RIDEs. For these reasons, various qualitative properties of solutions such as stability, US, UAS, instability, convergence, etc. of solutions of RIDEs attracted more and more attention of researchers. Now, there is a large number of works dealing with the investigation of these properties of solutions of RIDEs (see, in particular, [2–5, 11, 12, 18, 19, 22, 23, 25, 29, 30, 32, 33, 35, 37, 38, 40, 43–60] and the bibliography therein). However, to the authors' best knowledge, there are few works on the same concepts for nonlinear RIDEs with time-varying delays (see [19] and the bibliography therein). In reality, such qualitative properties of that kind of RIDEs can appear in many scientific problems representing processes whose speeds are defined not only by the present, but also by the previous states (for example, see [10, 11] and the bibliography therein). Therefore, it becomes apparent that an investigation of qualitative properties of RIDEs with time-varying delays is of importance.

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Now, we would like to summarize and recall some related studies on the qualitative properties of various RIDEs. Funakubo et al. [21] gave a necessary and sufficient condition for uniform asymptotic stability of the zero solution of the scalar linear integro-differential equation of Volterra type with constant delay

$$x'(t) = ax(t) - b\int_{t-h}^{t} x(s)\mathrm{d}s.$$

Here, the result on asymptotic stability was proved using root analysis of the characteristic equation associated to the integro-differential equation. Then, Matsunaga and Suzuki [31] obtained necessary and sufficient conditions for asymptotic stability of the trivial solution of the scalar system with constant delay

$$x'(t) = -ax(t) - b\int_{t-r}^{t} y(s)\mathrm{d}s, \quad y'(t) = -c\int_{t-r}^{t} x(s)\mathrm{d}s - ax(t).$$

The proof of the main result of [31] was given by an analysis of the locations of roots of the associated characteristic equation of this system. Later, Ngoc [36] dealt with the problem of exponential stability of linear differential systems of the form

$$x'(t) = A_0x(t) + \sum_{i=1}^{m} A_i x(t-h_i) + \int_{-h}^{0} B(s)x(t+s)\mathrm{d}s$$

and a similar problem for the same equation in the variable delay case. The proofs of the main results of [36] were based on spectral properties of Metzler matrices while the Lyapunov function method was utilized. Similarly, in Ngoc [34], some results on exponential stability of solutions of a certain linear integro-differential equation were obtained. Raffoul and Ünal [39] considered the linear integro-differential equation of Volterra type with constant delay

$$x'(t) = Px(t) - \int_{t-r}^{t} C(t,s)g(x(s))\mathrm{d}s.$$

By employing a Lyapunov functional, conditions for stability of the zero solution of this equation were obtained. Very recently, Tian et al. [42] were concerned with the delay-dependent stability analysis of linear systems with constant delay of the form

$$x'(t) = Ax(t) + Bx(t) + C\int_{t-h}^{t} x(s)\mathrm{d}s.$$

In [42], first, based on an integral equality, a new integral inequality was obtained. Then, to show the effectiveness of the new integral inequality, a new delay-dependent stability criterion was derived in terems of linear matrix inequalities for that integro-differential equation. Furthermore, exponential stability, asymptotic stability, Lyapunov stability and instability of solutions on the semi-axis in the Lebesgue sense, which presents the sort of suitable spaces of the right-hand sides and solutions, were studied in the monograph [6]. When we look at the classical monograph [28], the analysis concept is actually based on the use of the so-called monotone technique, which is an alternative to [11, Theorem 1]. The study methods of nonlinear systems on the basis of positivity of a resolvent operator for the linear part of the systems were presented in the monographs [26, 27]. The monotone technique, or, in other terminology, the positivity-based approach, was developed in the books [1, 20, 41] (see also [24, 36]). In these monographs, general linear operators acting from the space of continuous operators to the space of L^p, $1 \leq p \leq \infty$, were considered instead of the linear combination of deviation and integral operators, and only the properties of these operators were assumed. In the papers [7, 13, 14], positivity of resolvent operators to the linear system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x(t) + \sum_{k=1}^{N} B_k(t)x(t - h_k(t))$$

was obtained. This case allows to avoid the assumption on existence of zero delay in the matrix $A(t)$ with negative main diagonal. In this case, it is also very importatnt that a stabilization is achieved by the delay feedback control, and in the control devices of various technological models, delay naturally appears. For some conditions such as on the entries of the matrix $A(t)$, we refer the readers to [7–9, 13–16]. In the book of Agarwal et al. [1], the authors consider the system of functional differential equations

$$x_i'(t) + \sum_{j=1}^{n} (B_{ij}x_j)(t) = f_i(t), \quad t \in [0, \omega], \quad i = 1, 2, \ldots, n,$$

where $x = \mathrm{col}(x_1, x_2, \ldots, x_n)$, $B_{ij} : \mathrm{C}[0, \omega] \to \mathrm{L}_1[0, \omega]$ or $B_{ij} : \mathrm{C}[0, \omega] \to \mathrm{L}_\infty[0, \omega]$, $i, j = 1, 2, \ldots, n$, are linear continuous operators and $\mathrm{C}[0, \omega]$, $\mathrm{L}_1[0, \omega]$, and $\mathrm{L}_\infty[0, \omega]$ are the spaces of continuous, integrable, and essentiall bounded functions from $[0, \omega]$ to $\mathbb{R}$, respectively. It should be noted that the operators $B_{ij} : \mathrm{C} \to \mathrm{L}_\infty$ can be, for example, of the forms

$$\int_0^t y(s)\mathrm{d}_s r(t, s), \quad \int_0^t \Omega_{ij}(t, s)y(s)\mathrm{d}s, \quad \sigma(t)y(t - \tau(t)).$$

Their linear combinations and superpositions are also allowed. Agarwal et al. [1] proved an interesting and important result, [1, Theorem 16.3], with respect to the above system of functional differential equations. In [1, Theorem 16.3], it is assumed that the nondiagonal operators B_{ij}, $i, j = 1, 2, \ldots, n$, $i \neq j$, are negative. It is proved that for every bounded right-hand side f, the solution x is bounded on the semi-axis $[0, \infty)$. Moreover, the Cauchy matrix satisfies an exponential estimate, i.e., there exist positive constants α and N such that

$$0 \le C_{ij}(t, s) \le N \exp(-\alpha(t - s)), \quad 0 \le s \le t < \infty, \quad i, j = 1, 2, \ldots, n.$$

More interesting and related results are given in [1, Theorem 16.3]. Next, we should mention that in applications to stablization, namely the case when feedback delay control can stabilize the unstable system of ordinary differential equations

$$x_i'(t) = \sum_{j=1}^{n} a_{ij}x_j(t)$$

or integro-differential equations

$$x_i'(t) = \sum_{j=1}^{n} a_{ij}x_j(t) + \int_{t-\tau(t)}^{t} \Omega(t, s)x_j(s)\mathrm{d}s,$$

is interesting (see [17]). In our results concerning (1), there are consistencies with respect to the facts, but they are not of direct relevance to the results presented here. Next, we note that the nonlinear system of RIDEs (2) considered here is different from the integro-differential equations in [43–51]. In [43–51], some integro-differential equations have constant delay(s), some equations are without delay, some of them have scalar forms, and some other ones are in the form of systems. In the papers [43–51], the considered equations do not include any variable delay and terms such as $BF(x(t-\tau(t)))$ and $\int_{t-\tau(t)}^{t} \Omega(t,s)F(x(s))\mathrm{d}s$. This fact shows that the nonlinear system of RIDEs (2) improves the results in [43–51] for some particular cases. This is an improvement from the cases without delay and the cases with constant delay to the case of variable delay. Sometimes, system (2) can include the equations in [43–51] for some particular cases. The Lyapunov–Krasovskiĭ functional considered in this paper is different from those used in [43–51].

From this point, we recall a related work of Du [19]. In 1995, Du [19] considered the system of linear Volterra RIDEs

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + Bx(t-\tau(t)) + \int_{t-\tau(t)}^{t} \Omega(t,s)x(s)\mathrm{d}s, \tag{1}$$

in which $x \in \mathbb{R}^n$, $t \in [0,\infty)$, τ is a nonnegative and differentiable variable retardation, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$, and $\Omega \in \mathrm{C}(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^{n\times n})$. Du [19] constructed a Lyapunov–Krasovskiĭ functional for the system of RIDEs (1). Then, Du [19] proved a theorem on the UAS of the zero solution of the system of RIDEs (1) by help of that functional.

In this work, motivated by the system of RIDEs (1), the results of Du [19] and that in the bibliography of this paper, as an alternative to the system of RIDEs (1), we take into consideration a nonlinear system of RIDEs of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + BF(x(t-\tau(t))) + \int_{t-\tau(t)}^{t} \Omega(t,s)F(x(s))\mathrm{d}s + G(t,x), \tag{2}$$

i.e.,

$$\begin{aligned}\frac{\mathrm{d}x_i}{\mathrm{d}t} =& \sum_{j=1}^{n} a_{ij}(t)x_j + \sum_{j=1}^{n} b_{ij}F_j(x(t-\tau(t))) \\ &+ \sum_{j=1}^{n} \int_{t-\tau(t)}^{t} \Omega_{ij}(t,s)F_j(x(s))\mathrm{d}s + G_i(t,x),\end{aligned}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}^+ := [0,\infty)$, $\tau \in \mathrm{C}^1(\mathbb{R}^+,\mathbb{R}^+)$, $A = (a_{ij}) \in \mathrm{C}(\mathbb{R}^+,\mathbb{R}^{n\times n})$, and $\Omega = (\Omega_{ij}) \in \mathrm{C}(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^{n\times n})$, $B = (b_{ij}) \in \mathbb{R}^{n\times n}$, $F \in \mathrm{C}(\mathbb{R}^n,\mathbb{R}^n)$, $F(0) = 0$, and $G \in \mathrm{C}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$. In applications, typically linear nonsingular kernels Ω are used.

Let $x(t) = x(t,t_0,\phi)$ be a solution of the system of RIDEs (2) on $[t_0-\tau_0, t_0]$, $t_0 \geq 0$, $\max_{t\leq t_0} \tau(t) \leq \tau_0$, such that $x(t) = \phi(t)$ on $[t_0-\tau_0,t_0]$, where $\phi : [t_0-\tau_0,t_0] \to \mathbb{R}^n$ is a continuous initial function.

In this article, the US and UAS of zero solution, integrability and boundedness of solutions of the system of nonlinear RIDEs (2) are discussed when $G(t,x) = 0$. Further, boundedness of solutions at infinity is discussed when $G(t,x) \neq 0$. To reach the aim of this article, a suitable Lyapunov–Krasovskiĭ functional is defined to discuss these properties of solutions of the system of nonlinear RIDEs (2). The contributions of this article to the relevant literature can be explained as follows. It can be seen that the system of RIDEs (1) is linear. However, the system of RIDEs (2) has a nonlinear form. That is, the system of RIDEs (2) includes and improves the system of RIDEs (1). Thus, we extend and improve the results related to RIDEs (1) from the linear case to the nonlinear case such as (2). By this extension and improvement, this work has a contribution to the topic. Next, Du [19] discussed the UAS of the zero solution of the system of linear RIDEs (1). In addition to this qualitative property, we also study the integrability of solutions, boundedness of solutions, and boundedness of solutions at infinity of the system of nonlinear RIDEs (2). Moreover, in (1), A is a constant matrix, while in (2), A is allowed to depend on time. Finally, Du [19] did not give any example in a particular choice which satisfies the assumptions of [19, Theorem 2]. In this article, we present two specific examples, which confirm the applicability of the results of this article.

2. **Preliminaries.** We now begin with the following basic information. Let us consider the general nonautonomous retarded differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(t, x_t), \tag{3}$$

where $F : (-\infty, \infty) \times C \to \mathbb{R}^n$ is a continuous mapping with $F(t, 0) = 0$ and takes bounded sets into bounded sets. For some $r > 0$, $C = \mathrm{C}([-r, 0], \mathbb{R}^n)$ denotes the space of continuous functions $\phi : [-r, 0] \to \mathbb{R}^n$, $r > 0$. For any $a \geq 0$, some $t_0 \geq 0$, and $x \in \mathrm{C}([t_0 - r, t_0 + a], \mathbb{R}^n)$, we have $x_t = x(t + \phi)$ for $-r \leq \theta \leq 0$ and $t \geq t_0$. Let $x \in \mathbb{R}^n$. The norm $\|\cdot\|$ is defined by

$$\|x\| = \sum_{i=1}^{n} |x_i| \, .$$

Next, let $A \in \mathbb{R}^{n\times n}$. Then, $\|A\|$ is defined by

$$\|A\| = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|.$$

In this article, without loss of generality and mention, sometimes in place of $x(t)$, we will write x. For any $\phi \in C$, let

$$\|\phi\|_C = \sup_{\theta\in[-r,0]} \|\phi(\theta)\| = \|\phi(\theta)\|_{[-r,0]}$$

and

$$C_H = \{\phi : \ \phi \in C \quad \text{and} \quad \|\phi\|_C \leq H < \infty\} \, .$$

Throughout this article, it is assumed that the function F also fulfills the condition of the uniqueness of solutions of (3). It should be noted that the system of RIDEs (2) is a particular case of (3).

Assume that $x(t, t_0, \phi) = x(t)$ is a solution of (3) on $[t_0 - r, t_0]$, $t_0 \geq 0$ with $x(t) = \phi(t)$ for $t \in [t_0 - r, t_0]$, where $\phi : [t_0 - r, t_0] \to \mathbb{R}^n$ represents an initial continuous function. Let

$$V_1 : \mathbb{R}^+ \times C_H \to \mathbb{R}^+$$

be a continuous functional in t and ϕ with $V_1(t, 0) = 0$. Further, let $\frac{\mathrm{d}}{\mathrm{d}t}V_1(t, x)$ denote the derivative of $V_1(t, x)$ on the right through any solution x of (3).

Theorem 2.1 (Burton [10, Theorem 4.2.9])**.** *Assume that*

(A_1) *The functional $V_1(t, x)$ is locally Lipschitz in x, i.e., for every compact $S \subset \mathbb{R}^n$ and $\gamma > t_0$, there exists $K_{\gamma s} \in \mathbb{R}$ with $K_{\gamma s} > 0$ such that*

$$|V_1(t, x) - V_1(t, y)| \leq K_{\gamma s} \|x - y\|_{[t_0-r,t]}$$

for all $t \in [t_0, \gamma]$ and $x, y \in \mathrm{C}([t_0 - r, t_0], S)$.

(A_2) *Let $Z : \mathbb{R}^+ \times C_H \to \mathbb{R}^+$ be a functional that is one-sided locally Lipschitz in t, i.e.,*

$$|Z(t_2, \phi) - Z(t_1, \phi)| \leq K(t_2 - t_1), \ 0 < t_1 < t_2 < \infty, \quad K > 0, \quad \phi \in C_H.$$

(A_3) *There are four strictly increasing functions* $\omega, \omega_1, \omega_2, \omega_3 : \mathbb{R}^+ \to \mathbb{R}^+$ *with value* 0 *at* 0 *such that*

$$\omega(\|\phi(0)\|) + Z(t,\phi) \le V_1(t,\phi) \le \omega_1(\|\phi(0)\|) + Z(t,\phi),$$
$$Z(t,\phi) \le \omega_2(\|\phi\|_C),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} V_1(t,x) \le -\omega_3(\|x(t)\|)$$

whenever $t \in \mathbb{R}^+$ *and* $x \in C_H$.

Then, the trivial solution of (3) *is uniformly asymptotic stable.*

3. **UAS, integrability, boundedness.** In the system of RIDEs (2), let $G(t,x) = 0$, i.e., we consider

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + BF(x(t-\tau(t))) + \int_{t-\tau(t)}^{t} \Omega(t,s)F(x(s))\mathrm{d}s. \tag{4}$$

We make the following hypotheses to state our first result.

(C_1) There exist $k_0 \in (0,1]$ and $L > 0$ such that

$$F(0) = 0, \quad \|F(x) - F(y)\| \le k_0 \|x - y\| \quad \text{for all} \quad x, y \in \mathbb{R}^n,$$
$$\int_t^\infty \|\Omega(u,s)\| \,\mathrm{d}u \le L < \infty \quad \text{for all} \quad t_0 \le s \le t.$$

(C_2) There exist $d_i > 0$, $i = 1, \dots, n$, and $\delta, \eta, \mu > 0$ such that, for all $t \ge t_0$,

$$d_i a_{ii}(t) + \sum_{j=1, j\ne i}^{n} d_j |a_{ji}(t)| \le -\delta,$$
$$\delta - \mu k_0 \int_t^\infty \|\Omega(u,t)\| \,\mathrm{d}u \ge \eta,$$
$$\mu(1-\tau'(t)) \int_t^\infty \|\Omega(u, t-\tau(t))\| \,\mathrm{d}u \ge \frac{\beta_2}{k_0} \|B\|,$$
$$0 < \beta_2 \le \mu, \quad \text{where} \quad \beta_2 := \max\{d_i : \ i = 1, \dots, n\},$$

and

$$0 \le \tau(t) \le 1 - \frac{\beta_2 \|B\|}{k_0 L} + \tau(0) =: \tau^*.$$

Theorem 3.1. *The zero solution of the system of RIDEs* (4) *is uniformly asymptotically stable if hypotheses* (C_1) *and* (C_2) *hold. Furthermore, under the same hypotheses, the solutions of the system of RIDEs* (4) *are integrable in the sense of Lebesgue on the interval* $[0,\infty)$*. In particular, the solutions of the system of RIDEs* (4) *are bounded.*

Proof. Consider the Lyapunov–Krasovskiĭ functional $V_2 = V_2(t,x)$ defined by

$$V_2(t,x) := \|Dx(t)\| + \mu \int_{t-\tau(t)}^{t} \int_t^\infty \|\Omega(u,s)\| \,\mathrm{d}u \, \|F(x(s))\| \,\mathrm{d}s, \tag{5}$$

where $D = \mathrm{diag}[d_1, \dots, d_n]$ and $\mu, d_1, \dots, d_n > 0$. From this point, we see that the functional $V_2(t,x)$ satisfies the relations

$$V_2(t,0) = 0 \quad \text{and} \quad \|Dx(t)\| \le V_2(t,x).$$

Next, it follows that

$$\begin{aligned}
|V_2(t,x) - V_2(t,y)| \leq & \, |\|Dx(t)\| - \|Dy(t)\|| \\
& + \mu \left| \int_{t-\tau(t)}^{t} \int_{t}^{\infty} \|\Omega(u,s)\| \, \mathrm{d}u \, [\|F(x(s))\| - \|F(y(s))\|] \, \mathrm{d}s \right| \\
\leq & \, |\|Dx(t)\| - \|Dy(t)\|| \\
& + \mu \int_{t-\tau(t)}^{t} \int_{t}^{\infty} \|\Omega(u,s)\| \, \mathrm{d}u \, |\|F(x(s))\| - \|F(y(s))\|| \, \mathrm{d}s \\
\leq & \sum_{i=1}^{n} d_i \, |x_i(t) - y_i(t)| \\
& + \mu \int_{t-\tau(t)}^{t} \int_{t}^{\infty} \|\Omega(u,s)\| \, \mathrm{d}u \, \|F(x(s)) - F(y(s))\| \, \mathrm{d}s \\
\leq & \beta_2 \, \|x(t) - y(t)\| \\
& + \mu k_0 \int_{t-\tau(t)}^{t} \int_{t}^{\infty} \|\Omega(u,s)\| \, \mathrm{d}u \, \|x(s) - y(s)\| \, \mathrm{d}s \\
\leq & \beta_2 \, \|x(t) - y(t)\| + \mu k_0 L \tau(t) \sup_{t-\tau(t) \leq s \leq t} \|x(s) - y(s)\| \\
\leq & (\beta_2 + \mu k_0 L \tau^*) \sup_{t-\tau^* \leq s \leq t} \|x(s) - y(s)\| \\
= & K \sup_{t-\tau^* \leq s \leq t} \|x(s) - y(s)\| ,
\end{aligned}$$

where $K := \beta_2 + \mu k_0 L \tau^*$. As a consequence of this result, the functional $V_2(t,x)$ satisfies the local Lipschitz condition in x. Hence, (A_1) from Theorem 2.1 is satisfied. Let

$$\beta_1 := \min\{d_i : \; i = 1, \ldots, n\}$$

and

$$Z(t,x) := \mu \int_{t-\tau(t)}^{t} \int_{t}^{\infty} \|\Omega(u,s)\| \, \mathrm{d}u \, \|F(x(s))\| \, \mathrm{d}s.$$

Then, it therefore follows, using these statements, the functional V_2, and hypotheses (C_1) and (C_2), that

$$\begin{aligned}
Z(t_2,x) - Z(t_1,x) = & \mu \int_{t_2-\tau(t_2)}^{t_2} \int_{t_2}^{\infty} \|\Omega(u,s)\| \, \mathrm{d}u \, \|F(x(s))\| \, \mathrm{d}s \\
& - \mu \int_{t_1-\tau(t_1)}^{t_1} \int_{t_1}^{\infty} \|\Omega(u,s)\| \, \mathrm{d}u \, \|F(x(s))\| \, \mathrm{d}s.
\end{aligned}$$

For the next step, we add and subtract the term

$$\mu \int_{t_1-\tau(t_1)}^{t_2-\tau(t_2)} \int_{t_1}^{\infty} \|\Omega(u,s)\| \, \mathrm{d}u \, \|F(x(s))\| \, \mathrm{d}s$$

to the last equation. We then get

$$\begin{aligned}
Z(t_2,x) - Z(t_1,x) = & \mu \int_{t_2-\tau(t_2)}^{t_2} \int_{t_2}^{\infty} \|\Omega(u,s)\| \, \mathrm{d}u \, \|F(x(s))\| \, \mathrm{d}s \\
& + \mu \int_{t_1}^{t_2-\tau(t_2)} \int_{t_1}^{\infty} \|\Omega(u,s)\| \, \mathrm{d}u \, \|F(x(s))\| \, \mathrm{d}s
\end{aligned}$$

$$\begin{aligned}
&- \mu \int_{t_1-\tau(t_1)}^{t_2-\tau(t_2)} \int_{t_1}^{\infty} \|\Omega(u,s)\| \,\mathrm{d}u \, \|F(x(s))\| \,\mathrm{d}s \\
\leq & \mu \int_{t_2-\tau(t_2)}^{t_2} \int_{t_1}^{\infty} \|\Omega(u,s)\| \,\mathrm{d}u \, \|F(x(s))\| \,\mathrm{d}s \\
&+ \mu \int_{t_1}^{t_2-\tau(t_2)} \int_{t_1}^{\infty} \|\Omega(u,s)\| \,\mathrm{d}u \, \|F(x(s))\| \,\mathrm{d}s \\
= & \mu \int_{t_1}^{t_2} \int_{t_1}^{\infty} \|\Omega(u,s)\| \,\mathrm{d}u \, \|F(x(s))\| \,\mathrm{d}s \\
\leq & \mu k_0 L \int_{t_1}^{t_2} \|x(s)\| \,\mathrm{d}s \\
\leq & \mu k_0 L M (t_2 - t_1),
\end{aligned}$$

where

$$M := \sup_{t_1 \leq s \leq t_2} \|x(s)\| \qquad \text{and} \qquad 0 < t_1 < t_2 < \infty.$$

Hence, (A_2) from Theorem 2.1 is satisfied. Next,

$$\beta_1 \|x(t)\| + Z(t,x) \leq V_2(t,x) \leq \beta_2 \|x(t)\| + Z(t,x) \tag{6}$$

and

$$\begin{aligned}
Z(t,x) = & \mu \int_{t-\tau(t)}^{t} \int_{t}^{\infty} \|\Omega(u,s)\| \,\mathrm{d}u \, \|F(x(s))\| \,\mathrm{d}s \\
\leq & \mu L k_0 \int_{t-\tau(t)}^{t} \|x(s)\| \,\mathrm{d}s \\
\leq & \mu L k_0 \tau(t) \sup_{t-\tau(t) \leq s \leq t} \|x(s)\| \\
\leq & \mu \tau^* L k_0 \|x(s)\|_{[t-\tau^*, t]} .
\end{aligned}$$

Moreover, by the functional $V_2(t,x)$ in (5), differentiating gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} V_2^+(t,x) = & \sum_{i=1}^{n} d_i x_i'(t) \operatorname{sgn} x_i(t+0) + \mu \|F(x(t))\| \int_{t}^{\infty} \|\Omega(u,t)\| \,\mathrm{d}u \\
& - \mu(1-\tau'(t)) \|F(x(t-\tau(t)))\| \int_{t}^{\infty} \|\Omega(u,t-\tau(t))\| \,\mathrm{d}u \\
& - \mu \int_{t-\tau(t)}^{t} \|F(x(s))\| \, \|\Omega(t,s)\| \,\mathrm{d}s.
\end{aligned} \tag{7}$$

Using (4) and assumptions (C_1) and (C_2), we can now derive

$$\begin{aligned}
\sum_{i=1}^{n} d_i \operatorname{sgn} x_i(t+0) x_i'(t) \leq & \sum_{i=1}^{n} d_i a_{ii}(t) |x_i(t)| + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} d_j |a_{ji}(t)| \, |x_i(t)| \\
& + \beta_2 \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}| \, |F_j(x(t-\tau(t)))| \\
& + \int_{t-\tau(t)}^{t} \sum_{i=1}^{n} \sum_{j=1}^{n} d_i |\Omega_{ij}(t,s)| \, |F_j(x(s))| \,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{n}\left(d_i a_{ii}(t)+\sum_{j=1,j\neq i}^{n} d_j\left|a_{ji}(t)\right|\right)\left|x_i(t)\right| \\
&\quad + \beta_2 \|B\| \|F(x(t-\tau(t)))\| \\
&\quad + \beta_2 \int_{t-\tau(t)}^{t} \|F(x(s))\| \|\Omega(t,s)\| \,\mathrm{d}s \\
&\leq -\delta \|x(t)\| + \beta_2 \|F(x(t-\tau(t)))\| \|B\| \\
&\quad + \beta_2 \int_{t-\tau(t)}^{t} \|\Omega(t,s)\| \|F(x(s))\| \,\mathrm{d}s. \qquad (8)
\end{aligned}
$$

Thereby, a combination of (7), (8), (C_1), and (C_2) gives

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V_2^{+}(t,x) \leq & -\left(\delta-\mu k_0\int_{t}^{\infty}\|\Omega(u,t)\|\,\mathrm{d}u\right)\|x(t)\| \\
& -\mu(1-\tau'(t))\|F(x(t-\tau(t)))\|\int_{t}^{\infty}\|\Omega(u,t-\tau(t))\|\,\mathrm{d}u \\
& +\beta_2\|B\|\|F(x(t-\tau(t)))\|-\mu\int_{t-\tau(t)}^{t}\|\Omega(t,s)\|\|F(x(s))\|\,\mathrm{d}s \\
& +\beta_2\int_{t-\tau(t)}^{t}\|F(x(s))\|\|\Omega(t,s)\|\,\mathrm{d}s \\
\leq & \quad -\eta\|x(t)\|-\frac{\beta_2}{k_0}\|B\|\|F(x(t-\tau(t)))\| \\
& +\beta_2\|B\|\|F(x(t-\tau(t)))\|+(\beta_2-\mu)\int_{t-\tau(t)}^{t}\|\Omega(t,s)\|\|F(x(s))\|\,\mathrm{d}s \\
= & \quad -\eta\|x(t)\|+\beta_2\left(1-\frac{1}{k_0}\right)\|B\|\|F(x(t-\tau(t)))\| \\
& +(\beta_2-\mu)\int_{t-\tau(t)}^{t}\|\Omega(t,s)\|\|F(x(s))\|\,\mathrm{d}s \\
\leq & \quad -\eta\|x(t)\|+(\beta_2-\mu)\int_{t-\tau(t)}^{t}\|\Omega(t,s)\|\|F(x(s))\|\,\mathrm{d}s \\
\leq & \quad -\eta\|x(t)\|.
\end{aligned}
$$

Hence, (A_3) from Theorem 2.1 is satisfied. Altogether, by Theorem 2.1, the zero solution is therefore UAS. Next, integrating, we obtain

$$
V_2(t,x)-V_2(t_0,\phi(t_0)) \leq -\eta\int_{t_0}^{t}\|x(s)\|\,\mathrm{d}s \quad \text{for all} \quad t\geq t_0.
$$

This shows that

$$
\eta\int_{t_0}^{t}\|x(s)\|\,\mathrm{d}s \leq V_2(t_0,\phi(t_0))-V_2(t,x) \leq V_2(t_0,\phi(t_0)),
$$

from which it immediately follows that

$$
\int_{t_0}^{t}\|x(s)\|\,\mathrm{d}s \leq \frac{V_2(t_0,\phi(t_0))}{\eta} =: A_0 > 0.
$$

As a consequence, this shows that the solutions of the system of RIDEs (4) are integrable on $[t_0, \infty)$. Finally, from the above discussion, it easy to verify that

$$\frac{\mathrm{d}}{\mathrm{d}t} V_2^+(t, x) \leq 0.$$

An integration of this inequality gives

$$V_2(t, x) - V_2(t_0, \phi(t_0)) \leq 0 \quad \text{for all} \quad t \geq t_0.$$

Using this relation, $\|Dx(t)\| \leq V_2(t, x(t))$, and $\beta_1 = \min\{d_i\}$, we obtain

$$\beta_1 \|x(t)\| \leq \|Dx(t)\| \leq V_2(t, x(t)) \leq V_2(t_0, \phi(t_0)).$$

Therefore,

$$\|x(t)\| \leq \frac{V_2(t_0, \phi(t_0))}{\beta_1} =: B_0 > 0 \quad \text{for all} \quad t \geq t_0.$$

Hence, x is bounded on $[t_0, \infty)$. This completes the proof. □

Remark 1. Instead of the nonlinear system of RIDEs (4), we modify the system to be of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + B(t)x + CF(x(t - \tau(t))) + \int_{t-\tau(t)}^{t} \Omega(t, s)F(x(s))\mathrm{d}s, \qquad (4^*)$$

where $A = (a_{ij}) \in \mathrm{C}(\mathbb{R}^+, \mathbb{R}^{n\times n})$ $B = (b_{ij}) \in \mathrm{C}(\mathbb{R}^+, \mathbb{R}^{n\times n})$, $C = (c_{ij}) \in \mathbb{R}^{n\times n}$, and the other functions in (4^*) are defined in the same way as in (4) (see [42]). In addition to assumption (C_1), we assume

(C_2^*) There exist $d_i, \delta_i > 0$, $i = 1, \ldots, n$, and $\eta, \mu > 0$ such that, for all $t \geq t_0$,

$$a_{ii}(t) + b_{ii}(t) < 0, \quad d_i\left(a_{ii}(t) + b_{ii}(t)\right) + \sum_{j=1, j\neq i}^{n} d_j\left(|a_{ji}(t)| + |b_{ji}(t)|\right) \leq -\delta_i,$$

$$\delta - \mu k_0 \int_t^\infty \|\Omega(u, t)\| \,\mathrm{d}u \geq \eta, \quad \text{where} \quad \delta := \min\{\delta_i : \ i = 1, \ldots, n\},$$

$$\mu(1 - \tau'(t)) \int_t^\infty \|\Omega(u, t - \tau(t))\| \,\mathrm{d}u \geq \frac{\beta_2}{k_0} \|C\|,$$

$$0 < \beta_2 \leq \mu, \quad \text{where} \quad \beta_2 := \max\{d_i : \ i = 1, \ldots, n\},$$

and

$$0 \leq \tau(t) \leq 1 - \frac{\beta_2 \|C\|}{k_0 L} + \tau(0) =: \tau^*.$$

Moreover, by the exact same functional $V_2(t, x)$ as in (5), differentiating gives (7). Using (4^*) and assumptions (C_1) and (C_2^*), we now derive

$$\begin{aligned}
\sum_{i=1}^{n} d_i \operatorname{sgn} x_i(t+0) x_i'(t) &\leq \sum_{i=1}^{n} d_i a_{ii}(t) |x_i(t)| + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} d_j |a_{ji}(t)| |x_i(t)| \\
&+ \sum_{i=1}^{n} d_i b_{ii}(t) |x_i(t)| + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} d_j |b_{ji}(t)| |x_i(t)| \\
&+ \beta_2 \sum_{i=1}^{n} \sum_{j=1}^{n} |c_{ij}| |F_j(x(t - \tau(t)))| \\
&+ \int_{t-\tau(t)}^{t} \sum_{i=1}^{n} \sum_{j=1}^{n} d_i |\Omega_{ij}(t, s)| |F_j(x(s))| \,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{n}\left(d_i\left(a_{ii}(t)+b_{ii}(t)\right)+\sum_{j=1,j\neq i}^{n} d_j\left(|a_{ji}(t)|+|b_{ji}(t)|\right)\right)|x_i(t)| \\
&\quad + \beta_2\,\|C\|\,\|F(x(t-\tau(t)))\| \\
&\quad + \beta_2\int_{t-\tau(t)}^{t}\|F(x(s))\|\,\|\Omega(t,s)\|\,\mathrm{d}s \\
&\le -\sum_{i=1}^{n}\delta_i\,|x_i(t)| + \beta_2\,\|F(x(t-\tau(t)))\|\,\|C\| \\
&\quad + \beta_2\int_{t-\tau(t)}^{t}\|\Omega(t,s)\|\,\|F(x(s))\|\,\mathrm{d}s \\
&\le -\,\delta\,\|x(t)\| + \beta_2\,\|F(x(t-\tau(t)))\|\,\|C\| \\
&\quad + \beta_2\int_{t-\tau(t)}^{t}\|\Omega(t,s)\|\,\|F(x(s))\|\,\mathrm{d}s. \qquad (8^*)
\end{aligned}
$$

Thereby, a combination of (7), (8^*), (C_1), and (C_2^*) gives

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V_2^{+}(t,x) \le{}& -\left(\delta-\mu k_0\int_{t}^{\infty}\|\Omega(u,t)\|\,\mathrm{d}u\right)\|x(t)\| \\
& -\mu(1-\tau'(t))\,\|F(x(t-\tau(t)))\|\int_{t}^{\infty}\|\Omega(u,t-\tau(t))\|\,\mathrm{d}u \\
& +\beta_2\,\|C\|\,\|F(x(t-\tau(t)))\| - \mu\int_{t-\tau(t)}^{t}\|\Omega(t,s)\|\,\|F(x(s))\|\,\mathrm{d}s \\
& +\beta_2\int_{t-\tau(t)}^{t}\|F(x(s))\|\,\|\Omega(t,s)\|\,\mathrm{d}s \\
\le{}& -\eta\,\|x(t)\| - \frac{\beta_2}{k_0}\,\|C\|\,\|F(x(t-\tau(t)))\| \\
& +\beta_2\,\|C\|\,\|F(x(t-\tau(t)))\| + (\beta_2-\mu)\int_{t-\tau(t)}^{t}\|\Omega(t,s)\|\,\|F(x(s))\|\,\mathrm{d}s \\
={}& -\eta\,\|x(t)\| + \beta_2\left(1-\frac{1}{k_0}\right)\|C\|\,\|F(x(t-\tau(t)))\| \\
& +(\beta_2-\mu)\int_{t-\tau(t)}^{t}\|\Omega(t,s)\|\,\|F(x(s))\|\,\mathrm{d}s \\
\le{}& -\eta\,\|x(t)\| + (\beta_2-\mu)\int_{t-\tau(t)}^{t}\|\Omega(t,s)\|\,\|F(x(s))\|\,\mathrm{d}s \\
\le{}& -\eta\,\|x(t)\|.
\end{aligned}
$$

Hence, (A_3) from Theorem 2.1 is satisfied. Thus, by Theorem 2.1, the zero solution of (4^*) is UAS.

Example 1. Consider the system of nonlinear RIDEs

$$\begin{aligned}\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} &= \begin{pmatrix} -3-6e^{-2t} & -e^{-3t} \\ -e^{-3t} & -3-6e^{-3t} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &+ \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{16} \end{pmatrix}\begin{pmatrix} \sin x_1\left(t-\frac{1}{4}\right) \\ \sin x_2\left(t-\frac{1}{4}\right) \end{pmatrix} \\ &+ \int_{t-\frac{1}{4}}^{t} \begin{pmatrix} e^{-2t+s} & 0 \\ e^{-2t+s}\sin s & e^{-t+s+\frac{1}{4}}+2e^{-2t+s} \end{pmatrix}\begin{pmatrix} \sin x_1(s) \\ \sin x_2(s) \end{pmatrix} \mathrm{d}s,\end{aligned} \tag{9}$$

where $t \geq \frac{1}{4}$. If we compare the system of RIDEs (9) and the system of RIDEs (4), then we have the relations

$$A(t) = \begin{pmatrix} -3-6e^{-2t} & -e^{-3t} \\ -e^{-3t} & -3-6e^{-3t} \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix},$$

$$B = \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad \tau(t) \equiv \frac{1}{4},$$

$$\Omega(t,s) = \begin{pmatrix} e^{-2t+s} & 0 \\ e^{-2t+s}\sin s & e^{-t+s+\frac{1}{4}}+2e^{-2t+s} \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix},$$

and

$$F(x) = \begin{pmatrix} \sin x_1 \\ \sin x_2 \end{pmatrix} = \begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix}, \quad \text{where} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Pick

$$d_1 = d_2 = \mu = k_0 = 1, \quad L = 2\cosh\frac{1}{4}, \quad \delta = 3, \quad \eta = \frac{1}{2}$$

so that

$$\beta_1 = \beta_2 = 1.$$

Clearly, $F(0) = 0$. Next, using $|\sin z| \leq |z|$ for all $z \in \mathbb{R}$, some simple computations give

$$\begin{aligned}\|F(x)-F(y)\| &= \left\| \begin{pmatrix} \sin x_1 - \sin y_1 \\ \sin x_2 - \sin y_2 \end{pmatrix} \right\| \\ &= |\sin x_1 - \sin y_1| + |\sin x_2 - \sin y_2| \\ &= 2\left|\cos\left(\frac{x_1+y_1}{2}\right)\sin\left(\frac{x_1-y_1}{2}\right)\right| \\ &\quad + 2\left|\cos\left(\frac{x_2+y_2}{2}\right)\sin\left(\frac{x_2-y_2}{2}\right)\right| \\ &\leq 2\left(\left|\sin\left(\frac{x_1-y_1}{2}\right)\right| + \left|\sin\left(\frac{x_2-y_2}{2}\right)\right|\right) \\ &\leq 2\left(\left|\frac{x_1-y_1}{2}\right| + \left|\frac{x_2-y_2}{2}\right|\right) \\ &= \|x-y\| = k_0\|x-y\|.\end{aligned}$$

For the next step, consider the matrix $\Omega(t,s)$. We have

$$\begin{aligned}\|\Omega(t,s)\| &= \max_{1\leq j\leq 2}\sum_{i=1}^{2}|\Omega_{ij}| \\ &= \max\{e^{-2t+s} + e^{-2t+s}|\sin s|, e^{-t+s+\frac{1}{4}} + 2e^{-2t+s}\}\end{aligned}$$

$$=e^{-t+s+\frac{1}{4}} + 2e^{-2t+s}$$

since

$$e^{-2t+s} + e^{-2t+s}\left|\sin s\right| \le 2e^{-2t+s} < e^{-t+s+\frac{1}{4}} + 2e^{-2t+s} \quad \text{for all} \quad s,t \in \mathbb{R}.$$

Therefore,

$$\begin{aligned}\int_t^\infty \left\|\Omega(u,s)\right\| \mathrm{d}u = &\int_t^\infty e^{-u+s+\frac{1}{4}}\mathrm{d}u + 2\int_t^\infty e^{-2u+s}\mathrm{d}u \\ =&e^{-t+s+\frac{1}{4}} + e^{-2t+s} \quad \text{for all} \quad s,t \in \mathbb{R}.\end{aligned} \tag{10}$$

Now, by (10), for $s \le t$ and $t \ge \frac{1}{4}$, we have

$$\begin{aligned}\int_t^\infty \left\|\Omega(u,s)\right\| \mathrm{d}u =&e^{-t+s+\frac{1}{4}} + e^{-2t+s} \\ \le&e^{-t+t+\frac{1}{4}} + e^{-2t+t} = e^{\frac{1}{4}} + e^{-t} \\ \le&e^{\frac{1}{4}} + e^{-\frac{1}{4}} = 2\cosh\frac{1}{4} = L.\end{aligned}$$

Thus, (C_1) is satisfied. In view of the matrix $A(t)$, we can readily see that

$$d_i a_{ii}(t) + \sum_{j=1, j\ne i}^{2} d_j \left|a_{ji}(t)\right| = a_{ii}(t) + \sum_{j=1, j\ne i}^{2} \left|a_{ji}(t)\right| \le -3 = -\delta$$

since

$$a_{11}(t) + \left|a_{21}(t)\right| = -3 - 6e^{-2t} + e^{-3t} \le -3 = -\delta$$

and

$$a_{22}(t) + \left|a_{12}(t)\right| = -3 - 6e^{-3t} + e^{-3t} \le -3 = -\delta.$$

We may use (10) with $s = t$ to see that

$$\delta - \mu k_0 \int_t^\infty \left\|\Omega(u,t)\right\| \mathrm{d}u = 3 - \left(e^{\frac{1}{4}} + e^{-t}\right) \ge 3 - 2\cosh\frac{1}{4} > \frac{1}{2} = \eta$$

since $t \ge \frac{1}{4}$. Next, we get

$$\left\|B\right\| = \frac{1}{16}.$$

Moreover, since $\tau'(t) \equiv 0$ and using (10) with $s = t - \tau(t)$, we find

$$\mu(1-\tau'(t))\int_t^\infty \left\|\Omega(u, t-\tau(t))\right\| \mathrm{d}u = 1 + e^{-t-\frac{1}{4}} > 1 > \frac{1}{16} = \frac{\beta_2}{k_0}\left\|B\right\|.$$

Finally,

$$1 - \frac{\beta_2\left\|B\right\|}{k_0 L} + \tau(0) = \frac{1}{4} + 1 - \frac{1}{32\cosh\frac{1}{4}} > \frac{1}{4} = \tau(t) > 0.$$

Thus, (C_2) is satisfied. Now, using Theorem 3.1, the trivial solution of (9) is uniformly asymptotically stable, and the nonzero solutions of the system of RIDEs (9) are integrable and bounded. In Figures 1 and 2, the system of RIDEs (9) was solved and the orbits of the solutions x_1 and x_2 were drawn for different initial values. In Figure 1, we used initial conditions 1, 0.75, and -1 for $x_1(1/4)$. In these cases, we took $4t$, $3t$, and $-4t$ for $\phi(t)$. We used the fourth-order Runge–Kutta method to obtain Figures 1 and 2.

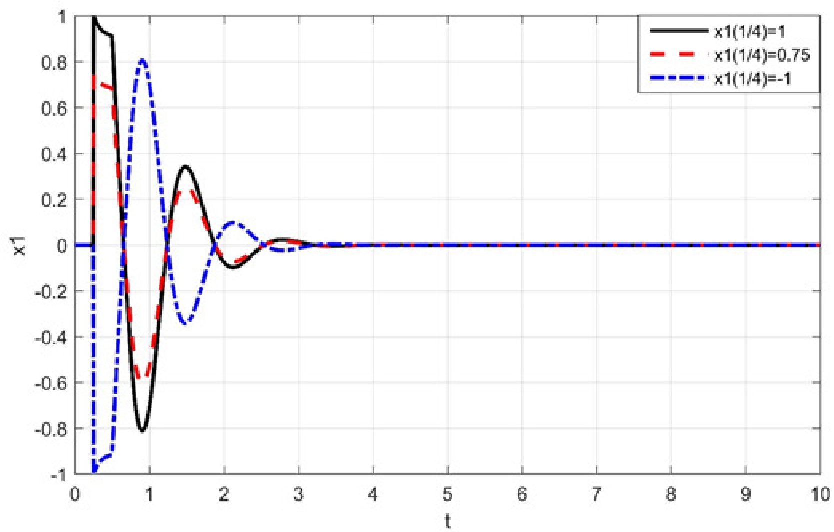

FIGURE 1. Solution x_1 of system of RIDEs (9) for different initial values.

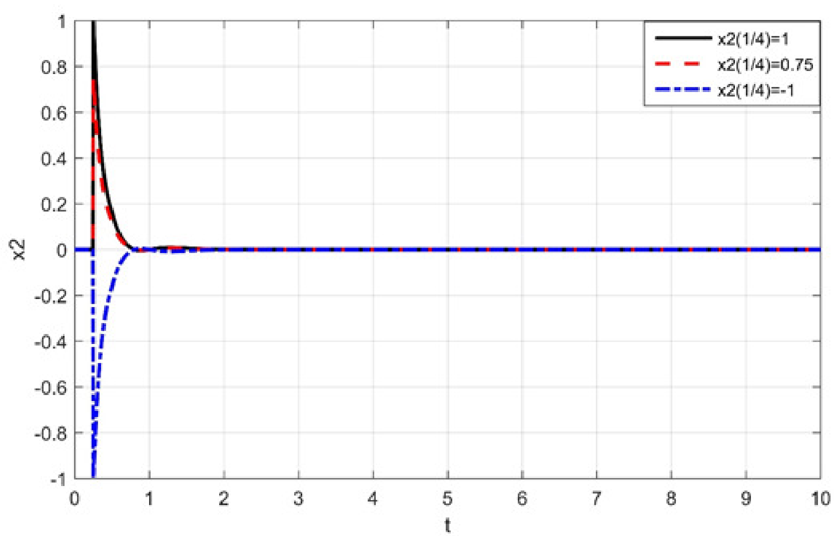

FIGURE 2. Solution x_2 of system of RIDEs (9) for different initial values.

4. **Boundedness at infinity.** In the system of RIDEs (2), let $G(t,x) \neq 0$. We make the following hypothesis to give our second result.

(C_3) Assume that there exists $g \in \mathrm{C}(\mathbb{R},\mathbb{R})$ such that

$$\|G(t,x)\| \le |g(t)|\,\|x\| \quad \text{for all} \quad t \ge t_0,\ x \in \mathbb{R}^n \quad \text{and} \quad \int_{t_0}^{\infty} |g(t)|\,\mathrm{d}t < \infty.$$

Theorem 4.1. *The solutions of the system of RIDEs (2) are bounded at infinity if hypotheses* (C_1)*–*(C_3) *hold.*

Proof. Choose the functional $V_2(t,x)$ as in (5). Then, in view of hypotheses (C_1)–(C_3) and the proof of Theorem 3.1, the time derivative of this functional along solutions of the system of RIDEs (2) is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_2^+(t,x) \leq& -\eta\,\|x\| + \left|\sum_{i=1}^{n} d_i \operatorname{sgn} x_i(t+0) G_i(t,x(t))\right| \\ \leq& \beta_2 \sum_{i=1}^{n} |G_i(t,x(t))| = \beta_2\,\|G(t,x(t))\| \\ \leq& \beta_2\,|g(t)|\,\|x(t)\| \leq \frac{\beta_2}{\beta_1}\,|g(t)|\,V_2(t,x),\end{aligned}$$

where we also have used (6). An integration of this inequality takes the form

$$\begin{aligned}V_2(t,x) \leq& V_2(t_0,\phi(t_0)) \exp\left(\frac{\beta_2}{\beta_1}\int_{t_0}^{t} |g(t)|\right)\mathrm{d}t \\ \leq& V_2(t_0,\phi(t_0)) \exp\left(\frac{\beta_2}{\beta_1}\int_{t_0}^{\infty} |g(t)|\right)\mathrm{d}t =: K_0.\end{aligned}$$

This, using again (6), means that

$$\|x(t)\| \leq \frac{K_0}{\beta_1} \quad \text{for all} \quad t \geq t_0.$$

Hence, x is bounded at infinity, and the proof is complete. □

Remark 2. In Theorem 4.1, the boundedness of solutions as $t \to \infty$ was proved without using the Gronwall inequality. By this fact, we have removed some unnecessary conditions and can obtain some boundedness results in the literature under less restrictive conditions (see [4, 11, 22, 29, 30, 43–48, 50] and the bibliography therein). Here, we would not like to state the details of these discussions.

Example 2. Consider the system of nonlinear RIDEs

$$\begin{aligned}\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} =& \begin{pmatrix} -3-6e^{-2t} & -e^{-3t} \\ -e^{-3t} & -3-6e^{-3t} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &+ \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{16} \end{pmatrix}\begin{pmatrix} \sin x_1\left(t-\frac{1}{4}\right) \\ \sin x_2\left(t-\frac{1}{4}\right) \end{pmatrix} \\ &+ \int_{t-\frac{1}{4}}^{t} \begin{pmatrix} e^{-2t+s} & 0 \\ e^{-2t+s}\sin s & e^{-t+s+\frac{1}{4}} + 2e^{-2t+s} \end{pmatrix}\begin{pmatrix} \sin x_1(s) \\ \sin x_2(s) \end{pmatrix}\mathrm{d}s \\ &+ \begin{pmatrix} \frac{2tx_1}{1+t^4+x_1^4} \\ \frac{2tx_2}{1+t^4+x_2^4} \end{pmatrix},\end{aligned} \tag{11}$$

where $t \geq \frac{1}{4}$. If we compare the system of RIDEs (11) and the system of RIDEs (2), then in addition to the same relations as in Example 1, we have

$$G(t,x) = \begin{pmatrix} \frac{2tx_1}{1+t^4+x_1^4} \\ \frac{2tx_2}{1+t^4+x_2^4} \end{pmatrix}.$$

Moreover, we do not need to show the satisfaction of hypotheses (C_1) and (C_2) as this already has been shown in Example 1. Clearly, for $t \geq 0$, we have

$$\|G(t,x)\| = \left\| \begin{pmatrix} \frac{2tx_1}{1+t^4+x_1^4} \\ \frac{2tx_2}{1+t^4+x_2^4} \end{pmatrix} \right\| = \frac{2t\,|x_1|}{1+t^4+x_1^4} + \frac{2t\,|x_2|}{1+t^4+x_2^4}$$

$$\leq \frac{2t\,|x_1|}{1+t^4} + \frac{2t\,|x_2|}{1+t^4} = \frac{2t}{1+t^4}\left[|x_1| + |x_2|\right] = |g(t)|\,\|x\|,$$

where

$$g(t) := \frac{2t}{1+t^4}.$$

Integrating the function $|g|$, we obtain

$$\int_0^\infty |g(t)|\,\mathrm{d}t = \int_0^\infty \frac{2t}{1+t^4}\mathrm{d}t = \frac{\pi}{2} < \infty.$$

Consequently, hypothesis (C_3) holds, and thus, by Theorem 4.1, the solutions of the system of RIDEs (11) are bounded as $t \to \infty$. In Figures 3 and 4, the system of RIDEs (11) was solved and the orbits of the solutions x_1 and x_2 were drawn for different initial values.

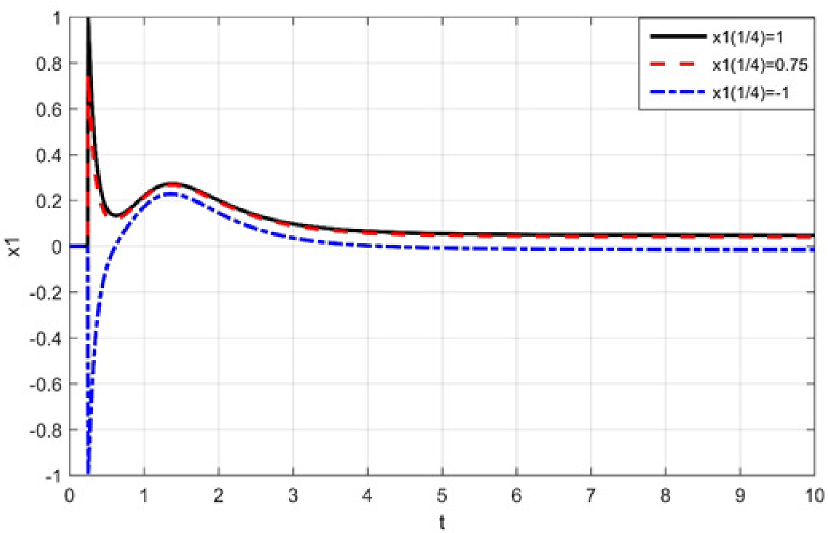

FIGURE 3. Bounded solution x_1 of system of RIDEs (11) for different initial values.

5. **Conclusion.** A system of nonlinear RIDEs with variable delay was considered. To the best of our information, firstly in the literature, the qualitative motion of solutions of this mathematical model of the RIDEs is investigated. Here, specifically, new and less restrictive sufficient conditions than those in the available literature for the stability, uniform asymptotic stability, boundedness, integrability, and boundedness of solutions at infinity were obtained. Those conditions were required by two new results, Theorems 3.1 and 4.1. Since the Gronwall inequality was not used in the results of this paper, Theorems 3.1 and 4.1, this fact allows weaker conditions than those that are available in the literature. From this point, we have

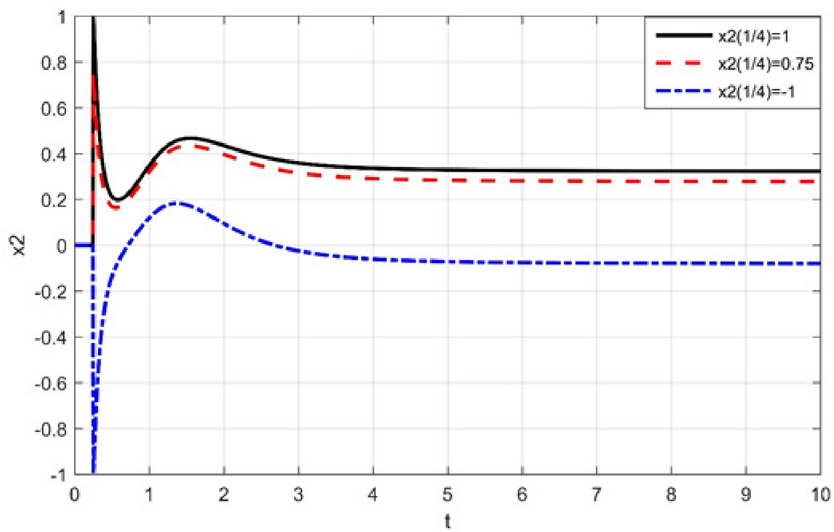

FIGURE 4. Bounded solution x_2 of system of RIDEs (11) for different initial values.

removed some unnecessary conditions related to some problems in the literature. Two examples are presented to verify the applicability of Theorems 3.1 and 4.1.

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REFERENCES

[1] R. P. Agarwal, L. Berezansky, E. Braverman and A. Domoshnitsky, *Nonoscillation Theory of Functional Differential Equations with Applications*, Springer, New York, 2012.

[2] R. P. Agarwal, M. Bohner, A. Domoshnitsky and Y. Goltser, Floquet theory and stability of nonlinear integro-differential equations, *Acta Math. Hungar.*, **109** (2005), 305–330.

[3] S. Ahmad and M. R. Mohana Rao, Stability of Volterra diffusion equations with time delays, *Appl. Math. Comput.*, **90** (1998), 143–154.

[4] F. Alahmadi, Y. N. Raffoul and S. Alharbi, Boundedness and stability of solutions of nonlinear Volterra integro-differential equations, *Adv. Dyn. Syst. Appl.*, **13** (2018), 19–31.

[5] J. A. D. Appleby and D. W. Reynolds, On necessary and sufficient conditions for exponential stability in linear Volterra integro-differential equations, *J. Integral Equations Appl.*, **16** (2004), 221–240.

[6] N. V. Azbelev and P. M. Simonov, *Stability of Differential Equations with Aftereffect*, vol. 20 of Stability and Control: Theory, Methods and Applications, Taylor & Francis, London, 2003.

[7] D. Baĭnov and A. Domoshnitsky, Nonnegativity of the Cauchy matrix and exponential stability of a neutral type system of functional-differential equations, *Extracta Math.*, **8** (1993), 75–82.

[8] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, vol. 9 of Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994, Revised reprint of the 1979 original.

[9] M. Bershadsky, M. V. Chirkov, A. Domoshnitsky, S. V. Rusakov and I. L. Volinsky, Distributed control and the Lyapunov characteristic exponents in the model of infectious diseases, *Complexity*, **2019** (2018), Art. ID 5234854, 12.

[10] T. A. Burton, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Dover Publications, Inc., Mineola, NY, 2005, Corrected version of the 1985 original.

[11] T. A. Burton, *Volterra Integral and Differential Equations*, vol. 202 of Mathematics in Science and Engineering, 2nd edition, Elsevier B. V., Amsterdam, 2005.

[12] C. Corduneanu and I. W. Sandberg (eds.), *Volterra Equations and Applications*, vol. 10 of Stability and Control: Theory, Methods and Applications, Gordon and Breach Science Publishers, Amsterdam, 2000, Papers from the Volterra Centennial Symposium held at the University of Texas, Arlington, TX, May 23–25, 1996.

[13] A. Domoshnitsky and E. Fridman, A positivity-based approach to delay-dependent stability of systems with large time-varying delays, *Systems Control Lett.*, **97** (2016), 139–148.

[14] A. Domoshnitsky, M. Gitman and R. Shklyar, Stability and estimate of solution to uncertain neutral delay systems, *Bound. Value Probl.*, **2014** (2014), 14pp.

[15] A. Domoshnitsky and R. Shklyar, Positivity for non-Metzler systems and its applications to stability of time-varying delay systems, *Systems Control Lett.*, **118** (2018), 44–51.

[16] A. Domoshnitsky, I. L. Volinsky and M. Bershadsky, Around the model of infection disease: The Cauchy matrix and its properties, *Symmetry*, **11** (2019), 1016.

[17] A. Domoshnitsky, I. L. Volinsky, A. Polonsky and A. Sitkin, Stabilization by delay distributed feedback control, *Math. Model. Nat. Phenom.*, **12** (2017), 91–105.

[18] X. T. Du, Stability of Volterra integro-differential equations with respect to part of the variables, *Hunan Ann. Math.*, **12** (1992), 110–115.

[19] X. T. Du, Some kinds of Liapunov functional in stability theory of RFDE, *Acta Math. Appl. Sinica (English Ser.)*, **11** (1995), 214–224.

[20] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and applications*, Pure and Applied Mathematics (New York), Wiley-Interscience, New York, 2000.

[21] M. Funakubo, T. Hara and S. Sakata, On the uniform asymptotic stability for a linear integro-differential equation of Volterra type, *J. Math. Anal. Appl.*, **324** (2006), 1036–1049.

[22] T. Furumochi and S. Matsuoka, Stability and boundedness in Volterra integro-differential equations, *Mem. Fac. Sci. Eng. Shimane Univ. Ser. B Math. Sci.*, **32** (1999), 25–40.

[23] K. Gopalsamy, A simple stability criterion for a linear system of neutral integro-differential equations, *Math. Proc. Cambridge Philos. Soc.*, **102** (1987), 149–162.

[24] W. M. Haddad and V. Chellaboina, Stability theory for nonnegative and compartmental dynamical systems with time delay, *Systems Control Lett.*, **51** (2004), 355–361.

[25] C. Jin and J. Luo, Stability of an integro-differential equation, *Comput. Math. Appl.*, **57** (2009), 1080–1088.

[26] I. T. Kiguradze, Boundary value problems for systems of ordinary differential equations, in *Current Problems in Mathematics. Newest results, Vol. 30 (Russian)*, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1987, 3–103, 204, Translated in *J. Soviet Math.*, **43** (1988), 2259–2339.

[27] I. T. Kiguradze and B. Půža, *Boundary Value Problems for Systems of Linear Functional Differential Equations*, vol. 12 of Folia Facultatis Scientiarium Naturalium Universitatis Masarykianae Brunensis. Mathematica, Masaryk University, Brno, 2003.

[28] M. A. Krasnosel′skiĭ, G. M. Vaĭnikko, P. P. Zabreĭko, Y. B. Rutitskii and V. Y. Stetsenko, *Approximate Solution of Operator Equations*, Wolters-Noordhoff Publishing, Groningen, 1972, Translated from the Russian by D. Louvish.

[29] V. Lakshmikantham and M. R. Mohana Rao, *Theory of Integro-Differential Equations*, vol. 1 of Stability and Control: Theory, Methods and Applications, Gordon and Breach Science Publishers, Lausanne, 1995.

[30] W. E. Mahfoud, Boundedness properties in Volterra integro-differential systems, *Proc. Amer. Math. Soc.*, **100** (1987), 37–45.

[31] H. Matsunaga and M. Suzuki, Effect of off-diagonal delay on the asymptotic stability for an integro-differential system, *Appl. Math. Lett.*, **25** (2012), 1744–1749.

[32] M. R. Mohana Rao and V. Raghavendra, Asymptotic stability properties of Volterra integro-differential equations, *Nonlinear Anal.*, **11** (1987), 475–480.

[33] M. R. Mohana Rao and P. Srinivas, Asymptotic behavior of solutions of Volterra integro-differential equations, *Proc. Amer. Math. Soc.*, **94** (1985), 55–60.

[34] P. H. A. Ngoc, Novel criteria for exponential stability of functional differential equations, *Proc. Amer. Math. Soc.*, **141** (2013), 3083–3091.

[35] P. H. A. Ngoc, On stability of a class of integro-differential equations, *Taiwanese J. Math.*, **17** (2013), 407–425.

[36] P. H. A. Ngoc, Stability of positive differential systems with delay, *IEEE Trans. Automat. Control*, **58** (2013), 203–209.
[37] Y. Raffoul, Exponential stability and instability in finite delay nonlinear Volterra integro-differential equations, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, **20** (2013), 95–106.
[38] Y. Raffoul and H. Rai, Uniform stability in nonlinear infinite delay Volterra integro-differential equations using Lyapunov functionals, *Nonauton. Dyn. Syst.*, **3** (2016), 14–23.
[39] Y. Raffoul and M. Ünal, Stability in nonlinear delay Volterra integro-differential systems, *J. Nonlinear Sci. Appl.*, **7** (2014), 422–428.
[40] M. Rahman, *Integral Equations and Their Applications*, WIT Press, Southampton, 2007.
[41] H. L. Smith, *Monotone Dynamical Systems*, vol. 41 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 1995, An introduction to the theory of competitive and cooperative systems.
[42] J. Tian, Z. Ren and S. Zhong, A new integral inequality and application to stability of time-delay systems, *Appl. Math. Lett.*, **101** (2020), 106058, 7pp.
[43] C. Tunç, Properties of solutions of Volterra integro-differential equations with delay, *Appl. Math. Inf. Sci.*, **10** (2016), 1775–1780.
[44] C. Tunç, Qualitative properties in nonlinear Volterra integro-differential equations with delay, *J. Taibah Univ. Sci.*, **11** (2017), 309–314.
[45] C. Tunç, Stability and boundedness in Volterra integro-differential equations with delay, *Dynam. Systems Appl.*, **26** (2017), 121–130.
[46] C. Tunç and O. Tunç, New qualitative criteria for solutions of Volterra integro-differential equations, *Arab. J. Basic Appl. Sci.*, **25** (2018), 158–165.
[47] C. Tunç and O. Tunç, New results on the stability, integrability and boundedness in Volterra integro-differential equations, *Bull. Comput. Appl. Math.*, **6** (2018), 41–58.
[48] C. Tunç and O. Tunç, On behaviours of functional Volterra integro-differential equations with multiple time-lags, *J. Taibah Univ. Sci.*, **12** (2018), 173–179.
[49] C. Tunç and O. Tunç, On the exponential study of solutions of Volterra integro-differential equations with time lag, *Electron. J. Math. Anal. Appl.*, **6** (2018), 253–265.
[50] C. Tunç and O. Tunç, A note on the qualitative analysis of Volterra integro-differential equations, *J. Taibah Univ. Sci.*, **13** (2019), 490–496.
[51] O. Tunç, On the qualitative analyses of integro-differential equations with constant time lag, *Appl. Math. Inf. Sci.*, **14** (2020), 57–63.
[52] J. Vanualailai and S.-i. Nakagiri, Stability of a system of Volterra integro-differential equations, *J. Math. Anal. Appl.*, **281** (2003), 602–619.
[53] K. Wang, Uniform asymptotic stability in functional-differential equations with infinite delay, *Ann. Differential Equations*, **9** (1993), 325–335.
[54] L. Wang and X. T. Du, The stability and boundedness of solutions of Volterra integro-differential equations, *Acta Math. Appl. Sinica*, **15** (1992), 260–268.
[55] Q. Wang, Asymptotic stability of functional-differential equations with infinite time-lag, *J. Huaqiao Univ. Nat. Sci. Ed.*, **19** (1998), 329–333.
[56] Q. Wang, The stability of a class of functional differential equations with infinite delays, *Ann. Differential Equations*, **16** (2000), 89–97.
[57] Z. C. Wang, Z. X. Li and J. H. Wu, Stability properties of solutions of linear Volterra integro-differential equations, *Tohoku Math. J. (2)*, **37** (1985), 455–462.
[58] A.-M. Wazwaz, *Linear and Nonlinear Integral Equations*, Higher Education Press, Beijing; Springer, Heidelberg, 2011, Methods and applications.
[59] P.-X. Weng, Asymptotic stability for a class of integro-differential equations with infinite delay, *Math. Appl. (Wuhan)*, **14** (2001), 22–27.
[60] Z. D. Zhang, Asymptotic stability of Volterra integro-differential equations, *J. Harbin Inst. Tech.*, 1990, 11–19.

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