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Sharp oscillation criteria for second-order neutral delay differential equations

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This paper is a continuation of a recent work by the authors on the oscillatory properties of second-order half-linear neutral delay differential equations. Providing a new apriori bound for a nonoscillatory solution, we present a new oscillation criterion, which essentially improves the existing ones. In a particular nonneutral case, the obtained oscillation constant is unimprovable.

KEYWORDS

delay, half-linear neutral differential equation, oscillation, second order

MSC CLASSIFICATION

34C10; 34K11

1 | INTRODUCTION

The aim of this work is to study oscillation of the second-order half-linear neutral delay differential equation

$$\left(r\left(z'\right)^{\alpha}\right)'(t)+q(t)x^{\alpha}(\sigma(t))=0,\quad t\geq t_0>0, \tag{1.1}$$

where $z(t)=x(t)+p(t)x(\tau(t))$. Without further mention, we will assume

- (H$_1$) $\alpha>0$ is a quotient of odd positive integers;
- (H$_2$) $r\in C([t_0,\infty),(0,\infty))$ satisfies
$$\pi(t_0):=\int_{t_0}^{\infty}r^{-1/\alpha}(s)\mathrm{d}s<\infty;$$
- (H$_3$) $\sigma,\ \tau\in C([t_0,\infty),\mathbb{R})$, $\sigma(t)\leq t$, and $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\sigma(t)=\infty$;
- (H$_4$) $p\in C([t_0,\infty),[0,\infty))$ and $q\in C([t_0,\infty),(0,\infty))$;
- (H$_5$) there exists a constant $p_0\in[0,1)$ such that

$$\begin{aligned} p_0 &\geq p(t)\frac{\pi(\tau(t))}{\pi(t)}\quad \text{for}\quad \tau(t)\leq t\\ p_0 &\geq p(t)\quad \text{for}\quad \tau(t)\geq t. \end{aligned} \tag{1.2}$$

Under a solution of (1.1), we mean a function $x \in C([t_a, \infty), \mathbb{R})$ with $t_a = \min\{\tau(t_b), \sigma(t_b)\}$, for some $t_b \geq t_0$, which has the property $r(z')^\alpha \in C^1([t_a, \infty), \mathbb{R})$ and satisfies (1.1) on $[t_b, \infty)$. We only consider those solutions of (1.1) that exist on some half-line $[t_b, \infty)$ and satisfy the condition

$$\sup\{|x(t)| : t_c \leq t < \infty\} > 0 \text{ for any } t_c \geq t_b.$$

As is customary, a solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory. The equation itself is termed oscillatory if all its solutions oscillate.

Beginning with the pioneering work of Fite,[1] the problem of determining oscillation criteria for particular functional differential equations has been and still is a very active research area. For a compact summary of known results, we refer the reader to the monographs by Agarwal et al.,[2-4] Győri and Ladas,[5] and Saker.[6] With regard to many indications of the importance of second-order neutral differential equations in the applications as well as the number of mathematical problems involved,[7] oscillation theory for such equations has undergone rapid development in the past decades.[8-20]

This paper is a continuation of our earlier work,[17] removing the restrictions (see[17], (H_3)) $\tau(t) \leq t$ and $\sigma'(t) \geq 0$. For the reader's convenience, we start with a brief summary of the ideas employed therein.

If x is an eventually positive solution of (1.1), then z is also positive and either strictly increasing or strictly decreasing, see Lemma 2 below. From these two possibilities, the case of z positive and decreasing is the important one because conditions for the nonexistence of such solutions already imply that solutions with z positive and increasing do not exist. This observation made in a previous study[17] allowed to simplify a number of related results.[8-16,20,21] To achieve not only a simpler but a qualitatively stronger result, we used, following a previous study,[8] a generalized Riccati substitution

$$w := \delta\left(\frac{r(z')^\alpha}{z^\alpha} + \frac{1}{\pi^\alpha}\right) > 0, \quad \text{where} \quad \delta \in C^1([t_0, \infty), (0, \infty)),$$

to eliminate the important class of positive solutions with z positive and decreasing. It turns out that the quality of such a criterion relies on the sharpness of the lower bound of the quantity $z(\sigma(t))/z(t)$ occurring in the resulting Riccati-type inequality. As noted in a previous study,[17] all the related results[8-16,20,21] used the estimate

$$\frac{z(\sigma(t))}{z(t)} \geq 1,$$

which in fact causes that the impact of the deviation σ is practically removed. In a previous study,[17] we obtained a sharper apriori estimate of nonoscillatory solutions of (1.1) based on asymptotic properties of solutions of certain first-order delay differential inequality.

Lemma 1 (See [17, Lemma 2.5]). *Assume that $\tau(t) \leq t$, $\sigma'(t) \geq 0$, and*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} \tilde{Q}(s)\mathrm{d}s \geq \rho$$

with

$$\tilde{Q}(t) = \left(\frac{1}{r(t)}\int_{t_1}^{t} q(s)\left(1 - p(\sigma(s))\frac{\pi(\tau(\sigma(s)))}{\pi(\sigma(s))}\right)^\alpha \mathrm{d}s\right)^{1/\alpha}$$

holds for some $\rho \in (0, 1/e]$. If x is a positive solution of (1.1) with $z > 0$ satisfying $z' < 0$ on $[t_1, \infty)$, then there exists a sufficiently large t_2 such that, for some $n \in \mathbb{N}_0$,

$$\frac{z(\sigma(t))}{z(t)} \geq f_n(\rho) \quad \text{for} \quad t \geq t_2,$$

where $f_n(\rho)$ is defined by

$$f_0(\rho) = 1, \quad f_{n+1}(\rho) = \mathrm{e}^{\rho f_n(\rho)}, \quad n \in \mathbb{N}_0. \tag{1.3}$$

In this work, employing a different technique, we will sequentially improve the lower bound of the quantity $z(\sigma(t))/z(t)$ up to its limit value. Our approach does not depend on the constant $1/e$ occurring in Lemma 1 due to a comparison with first-order equations and allows us to analytically compute resulting integrals in each iteration. The main result of the paper—the oscillation criterion for (1.1)—is a direct consequence of the obtained lower and upper bounds of a nonoscillatory solution. Moreover, our oscillation constant is unimprovable in the nonneutral case (see Example 1).

2 | MAIN RESULTS

In what follows, all occurring functional inequalities are assumed to hold eventually, that is, they are satisfied for all t large enough. As usual and without loss of generality, in the proofs of the main results, we only need to be concerned with positive solutions of (1.1) because the proofs for eventually negative solutions are similar.

Our results rely on a requirement of positive β_* defined by

$$\beta_* := \frac{1}{\alpha}\liminf_{t\to\infty} r^{1/\alpha}(t)\pi^{\alpha+1}(t)q(t).$$

Also, let us define

$$\lambda_* := \liminf_{t\to\infty}\frac{\pi(\sigma(t))}{\pi(t)}$$

and notice that $\lambda_* \geq 1$ due to (H_2) and (H_3). In our proofs, we will often use the fact that there exists a sufficiently large $t_* \geq t_0$ such that for arbitrary fixed $\beta \in (0, \beta_*)$ and $\lambda \in [1, \lambda_*)$,

$$q(t)r^{1/\alpha}(t)\pi^{\alpha+1}(t) \geq \alpha\beta \quad \text{and} \quad \frac{\pi(\sigma(t))}{\pi(t)} \geq \lambda \quad \text{on } [t_*, \infty). \tag{2.1}$$

For positive and finite β_* and λ_*, we define (as long as it exists) a sequence $\{\beta_n\}$ by

$$\begin{aligned}\beta_0 &:= (1-p_0)\sqrt[\alpha]{\beta_*},\\ \beta_{n+1} &:= \frac{\beta_0\lambda_*^{\beta_n}}{\sqrt[\alpha]{1-\beta_n}}, \quad n \in \mathbb{N}_0.\end{aligned}$$

By induction, it is easy to verify that if for some $n \in \mathbb{N}_0$, $\beta_i < 1$, $i = 0, 1, \ldots, n$, then β_{n+1} exists and

$$\beta_{n+1} = \ell_n\beta_n > \beta_n, \tag{2.2}$$

where

$$\begin{aligned}\ell_0 &:= \frac{\lambda_*^{\beta_0}}{\sqrt[\alpha]{1-\beta_0}},\\ \ell_{n+1} &:= \lambda_*^{\beta_0(\ell_n-1)}\sqrt[\alpha]{\frac{1-\beta_n}{1-\ell_n\beta_n}}, \quad n \in \mathbb{N}_0.\end{aligned} \tag{2.3}$$

Lemma 2. *Let $\beta_* > 0$. If x is an eventually positive solution of (1.1), then z eventually satisfies*

(i) $z > 0$, $\left(r(z')^\alpha\right)' < 0$, *and* $x(t) \geq z(t) - p(t)z(\tau(t))$;
(ii) $z' < 0$;
(iii) $(z/\pi)' \geq 0$;
(iv) $x \geq (1-p_0)z$;
(v) $\lim_{t\to\infty} z(t) = 0$.

Proof. Because x is an eventually positive solution of (1.1), there exists $t_1 \geq t_0$ such that

$$x(t) > 0, \quad x(\tau(t)) > 0, \quad \text{and} \quad x(\sigma(t)) > 0 \quad \text{for} \quad t \geq t_1.$$

(i) Obviously, for all $t \geq t_1$, $z(t) \geq x(t) > 0$, and $r(t)\big(z'(t)\big)^\alpha$ is decreasing and of one sign because

$$\big(r\big(z'\big)^\alpha\big)'(t) = -q(t)x^\alpha(\sigma(t)) < 0.$$

From the definition of z, we get

$$x(t) = z(t) - p(t)x(\tau(t)) \geq z(t) - p(t)z(\tau(t)), \quad t \geq t_1. \tag{2.4}$$

(ii) Assume on the contrary that $z' > 0$ for $t \geq t_1$. First, we show that

$$x(t) \geq kz(t), \quad t \geq t_2 \geq t_1, \tag{2.5}$$

where

$$k := \begin{cases} 1 - p_0, & \tau(t) \leq t, \\ 1 - \varepsilon p_0, & \tau(t) \geq t, \end{cases}$$

p_0 is defined by (1.2), and $\varepsilon \in (1, 1/p_0)$ is arbitrary but fixed.
If $\tau(t) \leq t$, (2.4) gives

$$x(t) \geq z(t)(1 - p(t)) \geq z(t)\left(1 - p(t)\frac{\pi(\tau(t))}{\pi(t)}\right) \geq z(t)(1 - p_0) > 0, \quad t \geq t_2,$$

where we used that z is increasing and π is decreasing.
Now consider the case $\tau(t) \geq t$. Using the monotonicity of $r^{1/\alpha}z'$, we have

$$z(t) = z(t_1) + \int_{t_1}^t r^{-1/\alpha}(s) \quad r^{1/\alpha}(s)z'(s)\mathrm{d}s \geq r^{1/\alpha}(t)z'(t)\int_{t_1}^t r^{-1/\alpha}(s)\mathrm{d}s = r^{1/\alpha}(t)z'(t)R(t, t_1),$$

where $R(t, t_1) = \int_{t_1}^t r^{-1/\alpha}(s)\mathrm{d}s$, and hence

$$\left(\frac{z}{R(\cdot, t_1)}\right)' = \frac{r^{1/\alpha}z'R(\cdot, t_1) - z}{r^{1/\alpha}R^2(\cdot, t_1)} \leq 0.$$

Using this monotonicity in (2.4), we get

$$x(t) \geq z(t)\left(1 - \frac{R(\tau(t), t_1)}{R(t, t_1)}p(t)\right).$$

Since in view of (H_2) and (H_3),

$$\lim_{t\to\infty}\frac{R(\tau(t), t_1)}{R(t, t_1)} = 1,$$

we have, for any $\varepsilon \in (1, 1/p_0)$ and $t_2 \geq t_1$ sufficiently large,

$$x(t) \geq z(t)\,(1 - \varepsilon p_0) > 0, \quad t \geq t_2.$$

Hence, (2.5) holds in either case. Now, using (2.5) in (1.1), we obtain

$$\big(r\big(z'\big)^\alpha\big)'(t) + k^\alpha q(t)z^\alpha(\sigma(t)) \leq 0, \quad t \geq t_2.$$

Integrating the above inequality from t_2 to t, and using that z is increasing, we find

$$\begin{aligned} r(t)\big(z'(t)\big)^{\alpha} &\le r(t_2)\big(z'(t_2)\big)^{\alpha} - k^{\alpha}\int_{t_2}^{t} q(s)z^{\alpha}(s)\mathrm{d}s \\ &\le r(t_2)\big(z'(t_2)\big)^{\alpha} - k^{\alpha}z^{\alpha}(t_2)\int_{t_2}^{t} q(s)\mathrm{d}s. \end{aligned} \tag{2.6}$$

For any $\beta \in (0, \beta_*)$ satisfying (2.1), there exists $t_3 \ge t_2$ such that

$$\int_{t_2}^{t} q(s)\mathrm{d}s \ge \beta \int_{t_2}^{t} \frac{\alpha}{r^{1/\alpha}(s)\pi^{\alpha+1}(s)}\mathrm{d}s = \beta\left[\frac{1}{\pi^{\alpha}(t)} - \frac{1}{\pi^{\alpha}(t_2)}\right], \quad t \ge t_3. \tag{2.7}$$

Using (2.7) along with the fact that $\lim_{t\to\infty}\pi(t) = 0$ in (2.6), we see that

$$r(t)\big(z'(t)\big)^{\alpha} \le r(t_2)\big(z'(t_2)\big)^{\alpha} - \beta z^{\alpha}(t_2)\left[\frac{1}{\pi^{\alpha}(t)} - \frac{1}{\pi^{\alpha}(t_2)}\right] \to -\infty \quad \text{as} \quad t \to \infty,$$

which contradicts the positivity of z'.

(iii) Since $r^{1/\alpha}z'$ is a negative and decreasing function, we have

$$z(t) \ge -\int_{t}^{\infty} r^{-1/\alpha}(s)r^{1/\alpha}(s)z'(s)\mathrm{d}s \ge -\pi(t)r^{1/\alpha}(t)z'(t). \tag{2.8}$$

Hence,

$$\left(\frac{z}{\pi}\right)' = \frac{r^{1/\alpha}z'\pi + z}{r^{1/\alpha}\pi^2} \ge 0.$$

(iv) This follows directly from (2.5) and (H_2).

(v) By (i) and (ii), z is positive and decreasing and so there exists a finite limit $\lim_{t\to\infty}z(t) = \ell \ge 0$. Assume on the contrary that $z(t) \ge \ell > 0$ eventually, say for $t \ge t_2 \ge t_1$. Consequently, (1.1) becomes

$$\big(r(z')^{\alpha}\big)'(t) \le -\ell^{\alpha}(1-p_0)^{\alpha}q(t), \quad t \ge t_2. \tag{2.9}$$

Integrating (2.9) from t_2 to t and using (2.7), we have, for any $\beta \in (0, \beta_*)$ and $t_3 \ge t_2$ large enough,

$$r(t)\big(z'(t)\big)^{\alpha} \le -\ell^{\alpha}(1-p_0)^{\alpha}\beta\left[\frac{1}{\pi^{\alpha}(t)} - \frac{1}{\pi^{\alpha}(t_2)}\right], \quad t \ge t_3,$$

that is,

$$z'(t) \le -\frac{\ell(1-p_0)\sqrt[\alpha]{\beta}}{r^{1/\alpha}(t)}\left[\frac{1}{\pi^{\alpha}(t)} - \frac{1}{\pi^{\alpha}(t_2)}\right]^{1/\alpha}.$$

Integrating the above inequality from t_3 to t, we obtain

$$\begin{aligned} z(t) &\le z(t_3) - \ell(1-p_0)\sqrt[\alpha]{\beta}\int_{t_3}^{t}\frac{1}{r^{1/\alpha}(s)}\left[\frac{1}{\pi^{\alpha}(s)} - \frac{1}{\pi^{\alpha}(t_2)}\right]^{1/\alpha}\mathrm{d}s \\ &\le z(t_3) - \ell(1-p_0)\sqrt[\alpha]{\beta}\left[1 - \frac{\pi^{\alpha}(t_3)}{\pi^{\alpha}(t_2)}\right]^{1/\alpha}\int_{t_3}^{t}\frac{\mathrm{d}s}{r^{1/\alpha}(s)\pi(s)} \\ &= z(t_3) - \ell(1-p_0)\sqrt[\alpha]{\beta}\left[1 - \frac{\pi^{\alpha}(t_3)}{\pi^{\alpha}(t_2)}\right]^{1/\alpha}\ln\frac{\pi(t_3)}{\pi(t)} \to -\infty \quad \text{as} \quad t \to \infty, \end{aligned}$$

a contradiction. Hence $\ell = 0$.

The proof is complete. □

Lemma 3. *Assume $\beta_* > 0$. If x is a positive solution of (1.1), then, for any $\varepsilon_0 \in (0,1)$,*

(i) $(z/\pi^{\varepsilon_0\beta_0})' < 0$ *eventually;*
(ii) $\varepsilon_0\beta_0 < 1$;
(iii) $\lambda_* < \infty$.

Proof. Pick $t_1 \geq t_0$ such that

$$x(t) > 0, \quad x(\tau(t)) > 0, \quad \text{and } x(\sigma(t)) > 0 \quad \text{for} \quad t \geq t_1,$$

z satisfies Lemma 2 for $t \geq t_1$, and (2.1) holds. Using Lemma 2 (iv) in (1.1), we have

$$\left(r\left(z'\right)^{\alpha}\right)'(t) + (1-p_0)^{\alpha} q(t) z^{\alpha}(\sigma(t)) \leq 0, \quad t \geq t_1,$$

which in view of (2.1) implies

$$\left(r\left(z'\right)^{\alpha}\right)'(t) + \frac{\beta\alpha(1-p_0)^{\alpha}}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)} z^{\alpha}(\sigma(t)) \leq 0. \tag{2.10}$$

(i) Integrating (2.10) from t_1 to t and using Lemma 2 (ii), we find

$$\begin{aligned} r(t)\left(z'(t)\right)^{\alpha} &\leq r(t_1)\left(z'(t_1)\right)^{\alpha} - \beta(1-p_0)^{\alpha} \int_{t_1}^{t} \frac{\alpha}{r^{1/\alpha}(s)\pi^{\alpha+1}(s)} z^{\alpha}(\sigma(s)) \mathrm{d}s \\ &\leq r(t_1)\left(z'(t_1)\right)^{\alpha} - \beta(1-p_0)^{\alpha} \int_{t_1}^{t} \frac{\alpha z^{\alpha}(s)}{r^{1/\alpha}(s)\pi^{\alpha+1}(s)} \mathrm{d}s \\ &\leq r(t_1)\left(z'(t_1)\right)^{\alpha} - \beta(1-p_0)^{\alpha} z^{\alpha}(t) \int_{t_1}^{t} \frac{\alpha}{r^{1/\alpha}(s)\pi^{\alpha+1}(s)} \mathrm{d}s \\ &= r(t_1)\left(z'(t_1)\right)^{\alpha} - \beta(1-p_0)^{\alpha} z^{\alpha}(t) \left(\frac{1}{\pi^{\alpha}(t)} - \frac{1}{\pi^{\alpha}(t_1)}\right). \end{aligned} \tag{2.11}$$

In view of Lemma 2 (v), there exists $t_2 \geq t_1$ such that

$$r(t_1)\left(z'(t_1)\right)^{\alpha} + \frac{\beta(1-p_0)^{\alpha} z^{\alpha}(t)}{\pi^{\alpha}(t_1)} < 0, \quad t \geq t_2.$$

Hence, using (2.11), we obtain

$$r\left(z'\right)^{\alpha} < -\beta(1-p_0)^{\alpha} \frac{z^{\alpha}}{\pi^{\alpha}},$$

that is,

$$\pi r^{1/\alpha} z' < -\sqrt[\alpha]{\beta}(1-p_0) z = -\varepsilon_0\beta_0 z,$$

where $\varepsilon_0 = \sqrt[\alpha]{\beta/\beta_*}$. Therefore,

$$\left(\frac{z}{\pi^{\varepsilon_0\beta_0}}\right)' = \frac{\pi r^{1/\alpha} z' + \varepsilon_0\beta_0 z}{r^{1/\alpha}\pi^{\varepsilon_0\beta_0+1}} < 0, \quad t \geq t_2.$$

(ii) The conclusion follows from the fact that $z/\pi^{\varepsilon_0\beta_0}$ is decreasing and z/π is increasing.
(iii) Assume that $\lambda_* = \infty$. In view of (i), for arbitrary but fixed $\lambda \in [1,\infty)$, there exists $t_3 \geq t_2$ such that

$$z(\sigma(t)) \geq \left(\frac{\pi(\sigma(t))}{\pi(t)}\right)^{\varepsilon_0\beta_0} z(t) \geq \lambda^{\varepsilon_0\beta_0} z(t), \quad t \geq t_3. \tag{2.12}$$

Let us choose λ such that

$$\lambda^{\varepsilon_0\beta_0} > \frac{1}{\varepsilon_0\beta_0}. \tag{2.13}$$

Integrating (2.10) from t_3 to t and using (2.12), we get

$$\begin{aligned} r(t)\big(z'(t)\big)^{\alpha} &\le r(t_3)\big(z'(t_3)\big)^{\alpha} - \beta(1-p_0)^{\alpha}\int_{t_3}^{t}\frac{\alpha}{r^{1/\alpha}(s)\pi^{\alpha+1}(s)}z^{\alpha}(\sigma(s))\mathrm{d}s \\ &\le r(t_3)\big(z'(t_3)\big)^{\alpha} - \lambda^{\alpha\varepsilon_0\beta_0}\beta(1-p_0)^{\alpha}z^{\alpha}(t)\int_{t_3}^{t}\frac{\alpha}{r^{1/\alpha}(s)\pi^{\alpha+1}(s)}\mathrm{d}s. \end{aligned}$$

Similarly as in part (i) of the proof, we arrive at

$$\pi r^{1/\alpha}z' < -\lambda^{\varepsilon_0\beta_0}\sqrt[\alpha]{\beta}(1-p_0)z = -\lambda^{\varepsilon_0\beta_0}\varepsilon_0\beta_0 z,$$

which by virtue of (2.13) implies

$$\pi r^{1/\alpha}z' < -z,$$

and therefore,

$$\left(\frac{z}{\pi}\right)' < 0,$$

which contradicts Lemma 2 (iii).

The proof is complete. □

Using Lemma 3, we have proved the following first main result of the paper.

Theorem 1. *Assume $\beta_* > 0$ and $\lambda_* = \infty$. Then (1.1) is oscillatory.*

Lemma 4. *Assume $\beta_* > 0$. If x is a positive solution of (1.1), then, for any $n \in \mathbb{N}_0$,*

$$\left(\frac{z}{\pi^{\beta_n}}\right)' < 0$$

eventually and

$$\beta_* \le \frac{\max\{k^{\alpha}(1-k)\lambda_*^{-\alpha k} : 0 < k < 1\}}{(1-p_0)^{\alpha}}. \tag{2.14}$$

Proof. Pick $t_1 \ge t_0$ such that

$$x(t) > 0, \quad x(\tau(t)) > 0, \quad \text{and} \quad x(\sigma(t)) > 0 \quad \text{for} \quad t \ge t_1,$$

z satisfies Lemma 2 for $t \ge t_1$, and (2.1) holds. The proof is divided into two parts.

First, we show by means of induction that for arbitrary $\varepsilon_n \in (0,1)$ and t large enough,

$$\pi r^{1/\alpha}z' < -\varepsilon_n\beta_n z,$$

which implies

$$\left(\frac{z}{\pi^{\varepsilon_n\beta_n}}\right)' < 0, \quad n \in \mathbb{N}_0, \tag{2.15}$$

where $\varepsilon_n \in (0,1)$ is defined by

$$\varepsilon_0 := \sqrt[\alpha]{\frac{\beta}{\beta_*}},$$
$$\varepsilon_{n+1} := \varepsilon_0 \sqrt[\alpha]{\frac{1-\beta_n}{1-\varepsilon_n\beta_n}\frac{\lambda^{\varepsilon_n\beta_n}}{\lambda_*^{\beta_n}}}, \quad n \in \mathbb{N}_0$$

for β and λ satisfying (2.1). The value of ε_n is arbitrary and depends on values of β and λ. It is easy to verify that

$$\lim_{(\beta,\lambda)\to(\beta_*,\lambda_*)} \varepsilon_n = 1.$$

Note that, in view of Lemma 2 (iii), (2.15) implies that

$$1-\varepsilon_n\beta_n > 0.$$

By Lemma 3 (i), (2.15) holds for $n = 0$. Next, assume that (2.15) holds for some $n > 0$ and $t \geq t_n \geq t_1$. Integrating (2.10) from t_n to t, we have

$$\begin{aligned} r(t)\left(z'(t)\right)^\alpha &\leq r(t_n)\left(z'(t_n)\right)^\alpha - \beta(1-p_0)^\alpha \int_{t_n}^t \frac{\alpha}{r^{1/\alpha}(s)\pi^{\alpha+1}(s)} z^\alpha(\sigma(s))\mathrm{d}s \\ &= r(t_n)\left(z'(t_n)\right)^\alpha - \beta(1-p_0)^\alpha \int_{t_n}^t \frac{\alpha\pi^{\alpha\varepsilon_n\beta_n}(\sigma(s))}{r^{1/\alpha}(s)\pi^{\alpha+1}(s)} \left(\frac{z}{\pi^{\varepsilon_n\beta_n}}\right)^\alpha(\sigma(s))\mathrm{d}s \\ &\leq r(t_n)\left(z'(t_n)\right)^\alpha - \beta(1-p_0)^\alpha \int_{t_n}^t \frac{\alpha\pi^{\alpha\varepsilon_n\beta_n}(\sigma(s))}{r^{1/\alpha}(s)\pi^{\alpha+1}(s)} \left(\frac{z}{\pi^{\varepsilon_n\beta_n}}\right)^\alpha(s)\mathrm{d}s \\ &\leq r(t_n)\left(z'(t_n)\right)^\alpha - \beta(1-p_0)^\alpha \left(\frac{z}{\pi^{\varepsilon_n\beta_n}}\right)^\alpha(t) \int_{t_n}^t \frac{\alpha\pi^{\alpha\varepsilon_n\beta_n}(\sigma(s))}{r^{1/\alpha}(s)\pi^{\alpha+1}(s)}\mathrm{d}s, \end{aligned}$$

where we used the induction hypothesis (2.15) twice in the last two inequalities. Using (2.1) in the last inequality, we arrive at

$$\begin{aligned} r(t)\left(z'(t)\right)^\alpha &\leq r(t_n)\left(z'(t_n)\right)^\alpha - \beta\lambda^{\alpha\varepsilon_n\beta_n}(1-p_0)^\alpha \left(\frac{z}{\pi^{\varepsilon_n\beta_n}}\right)^\alpha(t) \int_{t_n}^t \frac{\alpha}{r^{1/\alpha}(s)\pi^{\alpha(1-\varepsilon_n\beta_n)+1}(s)}\mathrm{d}s \\ &= r(t_n)\left(z'(t_n)\right)^\alpha \\ &\quad - \frac{\beta\lambda^{\alpha\varepsilon_n\beta_n}(1-p_0)^\alpha}{1-\varepsilon_n\beta_n}\left(\frac{z}{\pi^{\varepsilon_n\beta_n}}\right)^\alpha(t)\left(\frac{1}{\pi^{\alpha(1-\varepsilon_n\beta_n)}(t)} - \frac{1}{\pi^{\alpha(1-\varepsilon_n\beta_n)}(t_n)}\right). \end{aligned} \tag{2.16}$$

In view of (2.15), the function $z/\pi^{\varepsilon_n\beta_n}$ is bounded from above. We claim that

$$\lim_{t\to\infty} \frac{z(t)}{\pi^{\varepsilon_n\beta_n}(t)} = 0.$$

To prove the claim, it suffices to show that there is $\varepsilon > 0$ such that

$$\left(\frac{z}{\pi^{\varepsilon_n\beta_n+\varepsilon}}\right)' < 0. \tag{2.17}$$

Indeed, if

$$\lim_{t\to\infty} \frac{z(t)}{\pi^{\varepsilon_n\beta_n}(t)} = c > 0,$$

then

$$\frac{z(t)}{\pi^{\varepsilon_n\beta_n+\varepsilon}(t)} \geq \frac{c}{\pi^{\varepsilon}(t)} \to \infty \quad \text{as } t \to \infty, \tag{2.18}$$

which contradicts (2.17). Since $\lim_{t\to\infty}\pi(t) = 0$, there are

$$\ell \in \left(\lambda^{\beta_{n-1}(\varepsilon_{n-1}-1)} \sqrt[\alpha]{\frac{1-\beta_{n-1}}{1-\varepsilon_{n-1}\beta_{n-1}}}, 1 \right)$$

and $t_n' \geq t_n$ such that

$$\frac{1}{\pi^{\alpha(1-\varepsilon_n\beta_n)}(t)} - \frac{1}{\pi^{\alpha(1-\varepsilon_n\beta_n)}(t_n)} > \ell^{\alpha}\frac{1}{\pi^{\alpha(1-\varepsilon_n\beta_n)}(t)}, \quad t \geq t_n'.$$

Using the above estimate in (2.16), we obtain

$$r(t)\big(z'(t)\big)^{\alpha} < -\frac{\ell^{\alpha}\beta\lambda^{\alpha\varepsilon_n\beta_n}(1-p_0)^{\alpha}}{1-\varepsilon_n\beta_n}\left(\left(\frac{z}{\pi}\right)(t)\right)^{\alpha}, \quad t \geq t_n',$$

that is,

$$\pi r^{1/\alpha} z' < -\ell(1-p_0)\lambda^{\varepsilon_n\beta_n}\sqrt[\alpha]{\frac{\beta}{1-\varepsilon_n\beta_n}}z.$$

Simple computation shows that

$$\ell(1-p_0)\lambda^{\varepsilon_n\beta_n}\sqrt[\alpha]{\frac{\beta}{1-\varepsilon_n\beta_n}} - \varepsilon_n\beta_n = \varepsilon_0\beta_0\left(\frac{\ell\lambda^{\varepsilon_n\beta_n}}{\sqrt[\alpha]{1-\varepsilon_n\beta_n}} - \frac{\lambda^{\varepsilon_{n-1}\beta_{n-1}}}{\sqrt[\alpha]{1-\varepsilon_{n-1}\beta_{n-1}}}\right).$$

Since ε_n is arbitrarily large, in view of (2.2), we see that

$$\varepsilon_n\beta_n > \beta_{n-1}, \tag{2.19}$$

and hence

$$\ell(1-p_0)\lambda^{\varepsilon_n\beta_n}\sqrt[\alpha]{\frac{\beta}{1-\varepsilon_n\beta_n}} - \varepsilon_n\beta_n \geq \varepsilon_0\beta_0\left(\frac{\ell\lambda^{\beta_{n-1}}}{\sqrt[\alpha]{1-\beta_{n-1}}} - \frac{\lambda^{\varepsilon_{n-1}\beta_{n-1}}}{\sqrt[\alpha]{1-\varepsilon_{n-1}\beta_{n-1}}}\right) =: \varepsilon > 0.$$

Therefore,

$$\pi r^{1/\alpha} z' < -(\varepsilon_n\beta_n + \varepsilon)z,$$

and (2.17) holds. Hence, we prove the claim and so there exists $t_n'' \geq t_n'$ such that

$$r(t_n)\big(z'(t_n)\big)^{\alpha} + \frac{\beta\lambda^{\alpha\varepsilon_n\beta_n}(1-p_0)^{\alpha}}{1-\varepsilon_n\beta_n}\left(\frac{z}{\pi^{\varepsilon_n\beta_n}}\right)^{\alpha}(t)\frac{1}{\pi^{\alpha(1-\varepsilon_n\beta_n)}(t_n)} < 0, \ t \geq t_n''. \tag{2.20}$$

Using (2.20) in (2.16) implies that

$$\pi r^{1/\alpha} z' < -(1-p_0)\lambda^{\varepsilon_n\beta_n}\sqrt[\alpha]{\frac{\beta}{1-\varepsilon_n\beta_n}}z = -\varepsilon_{n+1}\beta_{n+1}z \tag{2.21}$$

and

$$\left(\frac{z}{\pi^{\varepsilon_{n+1}\beta_{n+1}}}\right)' < 0,$$

which completes the induction step.

Now, we are prepared to prove the assertion. Since, for some $n \in \mathbb{N}_0$, $\varepsilon_{n+1} \in (0,1)$ is arbitrarily large, we can assume that $\varepsilon_{n+1} < 1/\ell_n$, where ℓ_n is defined by (2.3). Using (2.19) in (2.21), we obtain

$$\pi r^{1/\alpha} z' < -\varepsilon_{n+1}\beta_{n+1} z < -\varepsilon_{n+1}\ell_n \beta_n z < -\beta_n z,$$

which immediately implies

$$\left(\frac{z}{\pi^{\beta_n}}\right)' < 0.$$

In view of Lemma 2 (iii), we deduce that

$$\beta_n < 1 \quad \text{for any} \ \ n \in \mathbb{N}_0.$$

This fact together with (2.2) means that the sequence $\{\beta_n\}$ is increasing and bounded above. Thus, there exists a finite limit $\lim_{n\to\infty}\beta_n = k$, where k is the smaller positive root of the indicial equation

$$\beta_*(1-p_0)^\alpha = k^\alpha(1-k)\lambda_*^{-\alpha k},$$

that necessarily implies (2.14). The proof is complete. □

Corollary 1. *Assume $\beta_* > 0$ and $\lambda_* < \infty$. Let x be an eventually positive solution of (1.1) with $z > 0$ satisfying $z' < 0$. Then for any $k \in (0,1)$,*

$$\frac{z(\sigma(t))}{z(t)} \geq k\lambda_*^{\beta_n}, \quad n \in \mathbb{N}_0. \tag{2.22}$$

Proof. It follows from Lemmas 2 (i) and 4 that $z > 0$ and $\left(z/\pi^{\beta_n}\right)' < 0$ eventually. Therefore,

$$\frac{z(\sigma(t))}{\pi^{\beta_n}(\sigma(t))} > \frac{z(t)}{\pi^{\beta_n}(t)}, \tag{2.23}$$

which in view of the definiton of λ_* implies (2.22). □

Using Lemma 4, we have proved the following second main result of the paper.

Theorem 2. *Assume $\lambda_* < \infty$. If*

$$\beta_* > \frac{\max\{k^\alpha(1-k)\lambda_*^{-\alpha k} : 0 < k < 1\}}{(1-p_0)^\alpha}, \tag{2.24}$$

then (1.1) is oscillatory.

Example 1. Consider the Euler type differential equation

$$\left(t^{\alpha+1}\left((x(t)+p_0 x(\lambda_1 t))'\right)^\alpha\right)' + q_0 x^\alpha(\lambda_2 t) = 0, \quad t \geq t_0 > 0, \tag{2.25}$$

where $\alpha > 0$ is a quotient of odd positive integers, $\lambda_1 > 0$, $\lambda_2 \in (0,1]$, $q_0 > 0$, and

$$p_0 < \begin{cases} \lambda_1^{1/\alpha} & \text{for} \quad \lambda_1 \leq 1, \\ 1 & \text{for} \quad \lambda_1 > 1. \end{cases}$$

Here,

$$\pi(t) = \frac{\alpha}{t^{1/\alpha}}, \quad \beta_* = q_0\alpha^\alpha, \quad \text{and} \quad \lambda_* = \frac{1}{\lambda_2^{1/\alpha}}.$$

It is easy to verify that (H_1)-(H_5) are satisfied. Hence, by Theorem 2, (2.25) is oscillatory if

$$\beta_* > \frac{\max\{k^\alpha(1-k)\lambda_*^{-\alpha k} : 0<k<1\}}{\left(1-p_0\lambda_1^{-1/\alpha}\right)^\alpha}. \tag{2.26}$$

To test the strength of the given result, we shall show when there exists a nonoscillatory solution of (2.25). It can be verified by a direct substitution into (2.25) that (2.25) has a nonoscillatory solution $x = t^m$ if $m \in (-1/\alpha, 0)$ satisfies the equation

$$-m^\alpha(m\alpha+1)\lambda_2^{-m\alpha}\left(1+p_0\lambda_1^m\right)^\alpha = q_0,$$

that is,

$$k^\alpha(1-k)\lambda_2^k\left(1+p_0\lambda_1^{-k/\alpha}\right)^\alpha = q_0\alpha^\alpha,$$

where we set $k = -m\alpha > 0$. In other words, (2.25) possesses a positive solution if

$$\beta_* \le \max\left\{k^\alpha(1-k)\lambda_*^{-\alpha k}\left(1+p_0\lambda_1^{-k/\alpha}\right)^\alpha : 0<k<1\right\}.$$

It is important to notice that in the nonneutral case, that is, if $p_0 = 0$, Theorem 2 provides a sharp result in a certain sense because (2.26) is necessary and sufficient for oscillation of the nonneutral Euler-type equation

$$\left(t^{\alpha+1}\left(x'(t)\right)^\alpha\right)' + q_0x^\alpha(\lambda_2 t) = 0.$$

Example 2. Now, consider

$$\left(t^{\alpha+1}\left((x(t)+p_0x(\lambda_1 t))'\right)^\alpha\right)' + q_0x^\alpha(t^{1/3}) = 0, \quad t \ge t_0 > 0, \tag{2.27}$$

with the same assumptions as for (2.25). Here, $\lambda_* = \infty$, and by Theorem 1, (2.27) is oscillatory.

In the next remark, we recall the existing criterion for (2.25) given in a previous study[17] for $\lambda_1 \le 1$. We have shown there that it improves related results from previous studies.[8-16,21]

Remark 1. By,[17, Theorem 2.4] if $\lambda_1 \le 1$ and

$$\rho := q_0^{1/\alpha}\left(1-p_0\lambda_1^{-1/\alpha}\right)\ln\left(\frac{1}{\lambda_2}\right) > \frac{1}{e}, \tag{2.28}$$

then (2.25) is oscillatory. If, however, (2.28) does not hold, the sequence $\{f_n\}$ defined by (1.3) has a finite limit, which can be expressed as

$$f(\rho) = \lim_{n\to\infty} f_n(\rho) = -\frac{W(-\rho)}{\rho},$$

where W standardly denotes the principal branch of the Lambert function.[22] By,[17, Corollary 2.8] we have that (2.25) is oscillatory if

$$q_0\left(1-p_0\lambda_1^{-1/\alpha}\right)^\alpha f(\rho) > \frac{1}{(\alpha+1)^{\alpha+1}} \tag{2.29}$$

or in terms of β_*, if

$$\beta_* > \frac{1}{\left(1-p_0\lambda_1^{-1/\alpha}\right)^\alpha f(\rho)}\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}. \tag{2.30}$$

Here, we notice that unlike (2.26), condition (2.30) is sharp only in the nondelayed nonneutral case, that is, when $\lambda_2 = 1$ and $p_0 = 0$. Then [17, Corollary 2.8] yields

$$q_0 > \frac{1}{(\alpha+1)^{\alpha+1}}, \tag{2.31}$$

which is a necessary and sufficient condition for oscillation of the half-linear ordinary Euler equation

$$\left(t^{\alpha+1}\left(x'(t)\right)^{\alpha}\right)' + q_0 x^{\alpha}(t) = 0.$$

We consider a particular case of (2.25) with $\lambda_1 = \lambda_2 = 1/2$, $\alpha = 5$, $p_0 = \lambda_1^{1/\alpha}/2$, and $q_0 = 0.0003904$. Then (2.29) reduces to

$$\beta_* := 1.22 \not> 1.9826,$$

however, (2.26) gives

$$\beta_* > 1.2100,$$

which implies that (2.25) is oscillatory.

Remark 2. Finally, we pose the open problem to show whether the condition

$$q_0\alpha^{\alpha} > \max\left\{k^{\alpha}(1-k)\lambda_2^{k}\left(1 + p_0\lambda_1^{-k/\alpha}\right)^{\alpha} : 0 < k < 1\right\}$$

is or is not sufficient for oscillation of (2.25).

CONFLICT OF INTEREST

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