CLOSED-FORM SOLUTIONS TO DISCRETE-TIME PORTFOLIO OPTIMIZATION PROBLEMS

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Abstract. In this paper, we study some discrete-time portfolio optimization problems. We introduce a discrete-time financial market model. The change in asset prices is modelled in contrast to the continuous-time market model by stochastic difference equations. We provide solutions of these stochastic difference equations. Then we introduce the discrete-time risk measures and the portfolio optimization problems. The main contributions of this paper are the closed-form solutions to the discrete-time portfolio models. For simulation purposes, the discrete-time financial market is often better suited. Several examples illustrating our theoretical results are provided.

1. Introduction

In [5, 6], the authors solved the continuous-time multi-period Earnings-at-Risk optimization problem

$$\left\{ \begin{array}{l} \displaystyle\min_{\varphi \in \mathbb{R}^n} \ \mathrm{EaR}(\varphi) \\ \text{s.t. } \mathbb{E}(X^{\varphi}(T)) \geq C \end{array} \right. \tag{1.1}$$

and the continuous-time multi-period mean-variance optimization problem

$$\left\{ \begin{array}{l} \displaystyle\min_{\varphi \in \mathbb{R}^n} \ \mathbb{V}\mathrm{ar}(\varphi) \\ \text{s.t. } \mathbb{E}(X^{\varphi}(T)) \geq C \end{array} \right. \tag{1.2}$$

with a constant rebalanced portfolio. A standard Black–Scholes financial market was assumed, which was modelled by stochastic differential equations (see [1, 4]). In this paper, we consider discrete-time versions of the problems (1.1) and (1.2). In Section 2, we briefly introduce the discrete-time financial market and the portfolio process. In Section 3, we prove some auxiliary results that are needed throughout the paper. Next, in Sections 4–7, we introduce several risk measures and solve the discrete-time one-period mean-Earnings-at-Risk problem, one-period Capital-at-Risk problem, one-period Value-at-Risk problem, and multi-period mean-variance problem.

2. Discrete-Time Financial Market

We construct our portfolio with $n+1$ assets. In our model we are considering discrete trading times on $[0, T] \cap \mathbb{N}_0$, where $T \in \mathbb{N}$. Let us denote the price of asset i at time t with $P_i(t)$ for $i = 0, \ldots, n$. We have one risk-free asset in our model. Without loss of generality it is asset $i = 0$. The risk-free asset is the bank account

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which pays constant interest with rate r every year. Denote by $P_0(t)$ the price of the risk-free asset at time t. Then P_0 follows the difference equation

$$P_0(t+1) - P_0(t) = P_0(t)r. \tag{2.1}$$

Lemma 2.1 (Solution of (2.1))**.** *The solution of* (2.1) *is given by*

$$P_0(t) = P_0(0)(1+r)^t, \quad t \in \mathbb{N}_0. \tag{2.2}$$

Proof. The relation (2.2) follows easily from (2.1) by induction. □

We now introduce the price processes of the risky assets. These are described by stochastic difference equations. First we need some notation to define the price processes of the risky assets. Let $b = (b_1, \ldots, b_n)'$ be the vector with the expected returns of the individual assets, and denote by $\sigma = (\sigma_{ij})_{1\le i,j\le n}$ the $n \times n$-matrix with the stock volatilities. To simplify the calculations, b and σ are assumed to be constant over the time. Now P_i follow the stochastic difference equations

$$P_i(t+1) - P_i(t) = P_i(t)\left(b_i + \sum_{j=1}^{n} \sigma_{ij}\left(B_j(t+1) - B_j(t)\right)\right), \quad i = 1, \ldots, n, \tag{2.3}$$

where $B(t)$ is a standard n-dimensional Brownian motion.

Lemma 2.2 (Solution of (2.3))**.** *The solution of* (2.3) *is given by*

$$P_i(t) = P_i(0)\prod_{a=0}^{t-1}\left(1 + b_i + \sum_{j=1}^{n} \sigma_{ij}\left(B_j(a+1) - B_j(a)\right)\right), \quad t \in \mathbb{N}_0. \tag{2.4}$$

Proof. The relation (2.4) follows easily from (2.3) by induction. □

Now we define the portfolio for our model. With $X^\varphi(t)$ we denote the total wealth at time t, and $\varphi_i(t)$ is the fraction of $X^\varphi(t)$ invested in asset i at time t. The vector $\varphi(t) = (\varphi_1(t), \ldots, \varphi_n(t))' \in \mathbb{R}^n$ is called the portfolio construction process, and $X^\varphi(t)$ is called the wealth process of the portfolio. In this paper we only consider so-called constant rebalanced investment portfolio strategies, i.e., $\varphi(t) \equiv \varphi$ is the same at each time $t \in [0, T] \cap \mathbb{N}_0$. We can calculate the weight of the risk-free asset in the portfolio by

$$\varphi_0 = 1 - \varphi'\mathbf{1}, \quad \text{where} \quad \mathbf{1} = (1, \ldots, 1)'.$$

If $\varphi_0 = 1$, then the entire wealth is invested in the risk-free asset ("pure-bond strategy"). The numbers of shares of the assets in our portfolio are

$$N_i(t) = X^\varphi(t)\frac{\varphi_i}{P_i(t)}, \quad i = 0, 1, \ldots, n. \tag{2.5}$$

Lemma 2.3 (Total wealth)**.** *The wealth of the portfolio at time t is given by*

$$X^\varphi(t) = \sum_{i=0}^{n} N_i(t)P_i(t), \quad t \in \mathbb{N}_0. \tag{2.6}$$

Proof. The calculation

$$\sum_{i=0}^{n} N_i(t)P_i(t) \overset{(2.5)}{=} \sum_{i=0}^{n} X^\varphi(t)\frac{\varphi_i}{P_i(t)}P_i(t) = X^\varphi(t)\sum_{i=0}^{n}\varphi_i = X^\varphi(t)$$

shows (2.6). □

The assumptions in this paper are: We have no transaction costs, no consumption over time, and a self-financing portfolio strategy. Now we find the change of portfolio wealth over one period. We obtain a stochastic difference equation.

Lemma 2.4 (Change in portfolio wealth over one period)**.** *We have*

$$X^{\varphi}(t+1) - X^{\varphi}(t) = X^{\varphi}(t)\left(r + \varphi'(b - r\mathbf{1}) + \varphi'\sigma(B(t+1) - B(t))\right). \tag{2.7}$$

Proof. Using our assumptions, we find

$$\begin{aligned}
X^{\varphi}(t+1) - X^{\varphi}(t) &\overset{(2.6)}{=} \sum_{i=0}^{n} N_i(t)(P_i(t+1) - P_i(t)) \\
&\overset{(2.1)}{\underset{}{\overset{(2.3)}{=}}} N_0(t) r P_0(t) + \sum_{i=1}^{n} N_i(t) b_i P_i(t) + \sum_{i=1}^{n} N_i(t) P_i(t) \sum_{j=1}^{n} \sigma_{ij}(B_j(t+1) - B_j(t)) \\
&\overset{(2.5)}{=} r X^{\varphi}(t)(1 - \varphi'\mathbf{1}) + \sum_{i=1}^{n} X^{\varphi}(t)\varphi_i b_i + \sum_{i=1}^{n} X^{\varphi}(t)\varphi_i \sum_{j=1}^{n} \sigma_{ij}(B_j(t+1) - B_j(t)) \\
&= X^{\varphi}(t)\left((1 - \varphi'\mathbf{1})r + \varphi' b + \varphi'\sigma(B(t+1) - B(t))\right).
\end{aligned}$$

This shows (2.7). □

Lemma 2.5 (Solution of (2.7))**.** *The solution of* (2.7) *is given by*

$$X^{\varphi}(t) = X^{\varphi}(0) \prod_{a=0}^{t-1} \left[1 + r + \varphi'(b - r\mathbf{1}) + \varphi'\sigma\Delta B(a)\right], \quad t \in \mathbb{N}_0. \tag{2.8}$$

Proof. The relation (2.8) follows easily from (2.7) by induction. □

We use the explicit formula (2.8) for $X^{\varphi}(t)$ to calculate expectation and variance of the portfolio. Some simple calculations using the properties of Brownian motion show the following results.

Theorem 2.6 (Expectation and variance of the wealth process)**.** *With*

$$\alpha := r + \varphi'(b - r\mathbf{1}), \quad c := \sigma'\varphi, \quad \text{and} \quad x := X^{\varphi}(0), \tag{2.9}$$

we have

$$X^{\varphi}(t) = x \prod_{a=0}^{t-1} \left[1 + \alpha + c'\Delta B(a)\right], \tag{2.10}$$

and therefore

$$\mathbb{E}(X^{\varphi}(t)) = x(1+\alpha)^t, \quad t \in \mathbb{N}_0 \tag{2.11}$$

and

$$\mathbb{V}\mathrm{ar}(X^{\varphi}(t)) = x^2\left[((1+\alpha)^2 + c'c)^t - (1+\alpha)^{2t}\right], \quad t \in \mathbb{N}_0. \tag{2.12}$$

Proof. By (2.9), (2.10) is the same as (2.8). We use (2.10) and the fact that increments of Brownian motion are independent with expectation zero to find

$$\mathbb{E}\left(X^{\varphi}(t)\right) = x \prod_{a=0}^{t-1} \mathbb{E}\left(1 + \alpha + \sum_{j=1}^{n} c_j \Delta B_j(a)\right) = x \prod_{a=0}^{t-1} (1+\alpha) = x(1+\alpha)^t.$$

This shows (2.11). Next, using (2.10) and the fact that increments of Brownian motion are independent with expectation zero and variance one, we find

$$\begin{aligned}\mathbb{E}((X^{\varphi}(t))^2) &= \mathbb{E}\left(x^2\prod_{a=0}^{t-1}\left[1+\alpha+c'\Delta B(a)\right]^2\right)\\ &= x^2\prod_{a=0}^{t-1}\mathbb{E}\left((1+\alpha)^2+2(1+\alpha)\sum_{j=1}^{n}c_j\Delta B_j(a)+\left(\sum_{j=1}^{n}c_j\Delta B_j(a)\right)^2\right)\\ &= x^2\prod_{a=0}^{t-1}\left((1+\alpha)^2+c'c\right)=x^2\left((1+\alpha)^2+c'c\right)^t.\end{aligned}$$

By (2.11) and $\mathbb{V}\mathrm{ar}(X^{\varphi}(t))=\mathbb{E}\left((X^{\varphi}(t))^2\right)-(\mathbb{E}(X^{\varphi}(t)))^2$, we get (2.12). □

We now introduce the main component of the risk measures used in this paper.

Definition 2.7. For a portfolio φ with wealth $X^{\varphi}(1)$, we define the risk measure $\mu(\varphi)$ corresponding to the β-quantile of $X^{\varphi}(1)$ by

$$\mathbb{P}(X^{\varphi}(1)\le\mu(\varphi))=\beta,\quad\text{where}\quad\beta\in(0,1).\tag{2.13}$$

In the next lemma we give an explicit expression for $\mu(\varphi)$ for a given β.

Lemma 2.8. *Let $\beta\in(0,1)$. If z_β denotes the β-quantile of the standard normal distribution, then $\mu(\varphi)$ in (2.13) is given by*

$$\mu(\varphi)=x(z_\beta\left\|\sigma'\varphi\right\|+1+r+\varphi'(b-r\mathbf{1})).\tag{2.14}$$

Proof. Since $X^{\varphi}(1)$ is standard normally distributed with expectation $x(1+\alpha)$ (see (2.11)) and variance $x^2c'c$ (see (2.12)), it follows that $z_\beta=(\mu(\varphi)-x(1+\alpha))/(x\sqrt{c'c})$, i.e., using (2.9), (2.14) holds. □

3. Auxiliary Results

In this section we provide some simple auxiliary results. For the rest of this paper we assume

$$\sigma\quad\text{is invertible and}\quad b\neq r\mathbf{1},\quad\text{and let}\quad\Theta:=\left\|\sigma^{-1}(b-r\mathbf{1})\right\|.\tag{3.1}$$

We first give the following three properties which are used often in Sections 4–7.

Lemma 3.1. *Assume* (3.1). *We have*

$$|\varphi'(b-r\mathbf{1})|\le\left\|\sigma'\varphi\right\|\Theta\quad\textit{for all}\quad\varphi\in\mathbb{R}^n.\tag{3.2}$$

Moreover, if we define

$$\varphi^*=\frac{\lambda(\sigma\sigma')^{-1}(b-r\mathbf{1})}{\Theta}\quad\textit{with}\quad\lambda\in\mathbb{R},$$

then we have

$$(\varphi^*)'(b-r\mathbf{1})=\lambda\Theta\tag{3.3}$$

and

$$\left\|\sigma'\varphi^*\right\|=|\lambda|.\tag{3.4}$$

Proof. First, we let $\varphi \in \mathbb{R}^n$ and use the Cauchy–Schwarz inequality to obtain

$$|\varphi'(b-r\mathbf{1})| = |(\sigma'\varphi)'(\sigma^{-1}(b-r\mathbf{1}))| \leq \|\sigma'\varphi\| \left\|\sigma^{-1}(b-r\mathbf{1})\right\| = \|\sigma'\varphi\|\,\Theta,$$

which shows (3.2). Next, we get

$$\begin{aligned}(\varphi^*)'(b-r\mathbf{1}) &= \lambda\frac{(b-r\mathbf{1})'(\sigma\sigma')^{-1}}{\Theta}(b-r\mathbf{1}) = \lambda\frac{(b-r\mathbf{1})'(\sigma')^{-1}\sigma^{-1}(b-r\mathbf{1})}{\Theta} \\ &= \lambda\frac{(\sigma^{-1}(b-r\mathbf{1}))'\sigma^{-1}(b-r\mathbf{1})}{\Theta} = \lambda\frac{\left\|\sigma^{-1}(b-r\mathbf{1})\right\|^2}{\Theta} = \lambda\Theta,\end{aligned}$$

which shows (3.3). Finally, we obtain

$$\|\sigma'\varphi^*\| = \left\|\sigma'\frac{\lambda(\sigma\sigma')^{-1}(b-r\mathbf{1})}{\Theta}\right\| = \left\|\lambda\frac{\sigma'(\sigma')^{-1}\sigma^{-1}(b-r\mathbf{1})}{\Theta}\right\| = \|\lambda\|\,\frac{\Theta}{\Theta} = |\lambda|,$$

which shows (3.4). □

Next we give a lemma that will be used frequently for the mean-CaR optimization problem (Section 5) and the mean-VaR optimization problem (Section 6). There and for the rest of this paper we assume

$$(3.5)\qquad 0 < \beta < \frac{1}{2}, \text{ and } z_\beta \text{ is the } \beta\text{-quantile of the standard normal distribution.}$$

Lemma 3.2. *Assume* (3.1) *and* (3.5). *Let $\Psi \in \mathbb{R}$ be independent of φ and let*

$$A := \{\varphi \in \mathbb{R}^n :\ \varphi'(b-r\mathbf{1}) + z_\beta\,\|\sigma'\varphi\| = \Psi\}.$$

If $\varphi \in A$, then

$$\varphi'(b-r\mathbf{1}) \geq \frac{\Psi\Theta}{\Theta - z_\beta} \tag{3.6}$$

and

$$(\Theta + z_\beta)\varphi'(b-r\mathbf{1}) \geq \Psi\Theta. \tag{3.7}$$

Proof. Note first that (3.5) implies $z_\beta < 0$. Let $\varphi \in A$. Then

$$\varphi'(b-r\mathbf{1}) \geq -|\varphi'(b-r\mathbf{1})| \overset{(3.2)}{\geq} -\|\sigma'\varphi\|\,\Theta \overset{(\varphi\in A)}{=} \frac{\Psi - \varphi'(b-r\mathbf{1})}{-z_\beta}\Theta,$$

i.e.,

$$-z_\beta\varphi'(b-r\mathbf{1}) \geq \Psi\Theta - \Theta\varphi'(b-r\mathbf{1}),$$

i.e.,

$$(\Theta - z_\beta)\varphi'(b-r\mathbf{1}) \geq \Psi\Theta,$$

which proves (3.6) since $\Theta - z_\beta > 0$. Next,

$$\varphi'(b-r\mathbf{1}) \leq |\varphi'(b-r\mathbf{1})| \overset{(3.2)}{\leq} \|\sigma'\varphi\|\,\Theta \overset{(\varphi\in A)}{=} \frac{\Psi - \varphi'(b-r\mathbf{1})}{z_\beta}\Theta,$$

i.e.,

$$z_\beta\varphi'(b-r\mathbf{1}) \geq \Psi\Theta - \Theta\varphi'(b-r\mathbf{1}),$$

which proves (3.7). □

Finally, we give a lemma which we use for the multi-period mean-variance problem (Section 7).

Lemma 3.3. *Let $c_1, c_2 \geq 0$ and $T \in \mathbb{N}$ and define $f : [0, \infty) \to \mathbb{R}$ by*

$$f(x) = \left((c_1 + x)^2 + c_2\right)^T - (c_1 + x)^{2T}.$$

Then f is increasing.

Proof. We let $x \geq 0$ and calculate

$$\begin{aligned} f'(x) &= T\left((c_1+x)^2 + c_2\right)^{T-1} 2(c_1+x) - 2T(c_1+x)^{2T-1} \\ &= 2T(c_1+x)\left[\left((c_1+x)^2+c_2\right)^{T-1} - (c_1+x)^{2T-2}\right] \\ &\geq 2T(c_1+x)\left[((c_1+x)^2)^{T-1} - (c_1+x)^{2T-2}\right] = 0, \end{aligned}$$

which completes the proof. □

4. One-Period Mean-Earnings-at-Risk Problem

In this section we introduce the discrete-time one-period mean-Earnings-at-Risk problem and provide a closed-form solution. The difference between the expected wealth after one period and the risk measure $\mu(\varphi)$ with the same portfolio φ is called *Earnings-at-Risk*.

Definition 4.1 (Earnings-at-Risk)**.** $\mathrm{EaR}(\varphi) := \mathbb{E}(X^\varphi(1)) - \mu(\varphi)$.

We solve the optimization problem

$$\left\{ \begin{array}{l} \displaystyle\min_{\varphi \in \mathbb{R}^n} \ \mathrm{EaR}(\varphi) \\ \text{s.t. } \mathbb{E}(X^\varphi(1)) \geq C, \end{array} \right. \tag{4.1}$$

where $C \in \mathbb{R}$ is the expected terminal wealth at time $T = 1$.

Theorem 4.2 (Closed-form solution of the discrete-time one-period mean-EaR optimization problem)**.** *Assume* (3.1) *and* (3.5)*. The closed-form solution of the one-period mean-Earnings-at-Risk problem* (4.1) *is given by*

$$\varphi_* = \frac{\lambda(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\Theta} \quad \textit{with} \quad \lambda = \frac{\left(\frac{C}{x} - 1 - r\right)^+}{\Theta},$$

where

$$z^+ = \left\{ \begin{array}{lll} z & \textit{if} & z \geq 0 \\ 0 & \textit{if} & z < 0 \end{array} \right. \quad \textit{for any} \quad z \in \mathbb{R}.$$

The expected wealth after one period is C with Earnings-at-Risk $-xz_\beta\lambda$.

Proof. Using (2.11) for $t = 1$ and (2.14), it suffices to show that $\varphi_* \in A$ and

$$g(\varphi) \geq g(\varphi_*) = -xz_\beta\lambda \quad \text{for all} \quad \varphi \in A,$$

where

$$g(\varphi) := -xz_\beta \left\|\sigma'\varphi\right\| \quad \text{and} \quad A := \left\{\varphi \in \mathbb{R}^n : \ \varphi'(b - r\mathbf{1}) \geq \frac{C}{x} - 1 - r\right\}.$$

To show this, first note that

$$(\varphi_*)'(b - r\mathbf{1}) \overset{(3.3)}{=} \lambda\Theta = \frac{\left(\frac{C}{x} - 1 - r\right)^+}{\Theta}\Theta = \left(\frac{C}{x} - 1 - r\right)^+ \geq \frac{C}{x} - 1 - r$$

implies $\varphi_* \in A$. Next, if $\varphi \in A$, then

$$\begin{aligned} g(\varphi) &= \frac{-xz_\beta}{\Theta} \|\sigma'\varphi\| \,\Theta \overset{(3.2)}{\geq} \frac{-xz_\beta}{\Theta} |\varphi'(b - r\mathbf{1})| \overset{(\varphi\in A)}{\geq} \frac{-xz_\beta}{\Theta}\left(\frac{C}{x} - 1 - r\right)^+ \\ &= -xz_\beta\lambda \overset{(3.4)}{=} -xz_\beta \|\sigma'\varphi_*\| = g(\varphi_*). \end{aligned}$$

This completes the proof. □

As an immediate consequence of Theorem 4.2 we get that the optimal Earnings-at-Risk is a function of the expected terminal wealth. An investor is now able to plot a graph for different expected terminal wealths. Since the supremum of EaR is infinity and the constraint of (4.1) is unbounded from above, the solution of the corresponding maximum problem is infinity. We denote with $\omega := \mathbb{E}(X^\varphi(1))$ the expected wealth after 1 year. We plug ω into λ given by Theorem 4.2 and get

$$\mathrm{EaR}(\omega) = -xz_\beta\lambda = -xz_\beta \frac{\left(\frac{\omega}{x} - 1 - r\right)^+}{\|\sigma^{-1}(b - r\mathbf{1})\|}.$$

Example 4.3. Let

$$r = 0.05, \quad b = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.3 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}$$

and

$$x = 1000, \quad C = 1056, \quad z_\beta = -1.64.$$

Now we calculate

$$\lambda = \frac{\left(\frac{C}{x} - 1 - r\right)^+}{\|\sigma^{-1}(b - r\mathbf{1})\|} = \frac{\frac{1056}{1000} - 1 - 0.05}{\left\| \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right\|} \approx 0.002384.$$

With that λ we calculate the Earnings-at-Risk for our portfolio with an expected terminal wealth of C as

$$\mathrm{EaR}(\varphi_*) = -xz_\beta\lambda = -1000 \cdot (-1.64) \cdot \lambda \approx 3.908947.$$

This is the minimal Earnings-at-Risk for the portfolio with an expected terminal wealth of 1056 at time 1. By Theorem 4.2, the optimal policy is given by

$$\varphi_* = \frac{\lambda \cdot \begin{pmatrix} 0.041 & 0.0242 & 0.0133 \\ 0.0242 & 0.1016 & 0.018 \\ 0.0133 & 0.018 & 0.0134 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix}}{\left\| \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right\|} \approx \begin{pmatrix} -0.006349 \\ -0.001753 \\ 0.026322 \end{pmatrix}.$$

This means 2.6322% are invested in asset 3 and the rest is invested risk free. Now we check if the expected wealth at time 1 really is 1056:

$$\mathbb{E}(X^\varphi(1)) = 1000 \left(1 + 0.05 + (\varphi_*)' \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right) = 1056.$$

5. One-Period Capital-at-Risk Problem

In this section we introduce the discrete-time one-period mean-Capital-at-Risk problem and provide a closed-form solution. The solution of the continuous-time optimization problem can be found in [3]. The difference between the possible risk-free profit after one period and the risk measure $\mu(\varphi)$ is called *Capital-at-Risk*.

Definition 5.1 (Capital-at-Risk)**.** $\mathrm{CaR}(\varphi) := x(1+r) - \mu(\varphi)$.

We accept a certain amount as Capital-at-Risk and we want to maximize the expected return. We solve the optimization wealth

$$\left\{\begin{array}{l}\max\limits_{\varphi\in\mathbb{R}^n} \mathbb{E}(X^\varphi(1)) \\ \text{s.t. } \mathrm{CaR}(\varphi) = C,\end{array}\right. \tag{5.1}$$

and we also solve the problem

$$\left\{\begin{array}{l}\min\limits_{\varphi\in\mathbb{R}^n} \mathbb{E}(X^\varphi(1)) \\ \text{s.t. } \mathrm{CaR}(\varphi) = C,\end{array}\right. \tag{5.2}$$

where C is the CaR at time $T=1$. An overview of the results given in this section can be found in Table 1.

Table 1. Overview mean-Capital-at-Risk problem

$\Theta + z_\beta$	C	Result	See
< 0	> 0	Found max and min	Theorem 5.2
> 0	> 0	Found min	Theorem 5.3
> 0	< 0	Found min	Theorem 5.4
< 0	< 0	$A = \emptyset$	Theorem 5.5

Theorem 5.2 (Closed-form solution to the discrete-time one-period mean-CaR optimization problem, part 1)**.** *Assume* (3.1), (3.5), *and*

$$\Theta + z_\beta < 0 \quad \textit{and} \quad C > 0.$$

The closed-form solution of the one-period mean-Capital-at-Risk problem (5.1) *is given by*

$$\varphi^* = \frac{\lambda(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\Theta} \quad \textit{with} \quad \lambda = -\frac{\frac{C}{x}}{\Theta + z_\beta}.$$

The closed-form solution of problem (5.2) *is given by*

$$\varphi_* = \frac{\mu(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\Theta} \quad \textit{with} \quad \mu = -\frac{\frac{C}{x}}{\Theta - z_\beta}.$$

The corresponding expected wealth after one period is

$$\mathbb{E}(X^{\varphi^*}(1)) = x\left(1 + r + \lambda\Theta\right) \quad \textit{and} \quad \mathbb{E}(X^{\varphi_*}(1)) = x\left(1 + r + \mu\Theta\right),$$

respectively, with $\mathrm{CaR}(\varphi^*) = \mathrm{CaR}(\varphi_*) = C$.

Proof. Using (2.11) for $t = 1$ and (2.14), it suffices to show that $\varphi^*, \varphi_* \in A$ and

$$x(1 + r + \mu\Theta) = g(\varphi_*) \leq g(\varphi) \leq g(\varphi^*) = x(1 + r + \lambda\Theta) \quad \text{for all} \quad \varphi \in A,$$

where

$$g(\varphi) := x(1{+}r{+}\varphi'(b{-}r\mathbf{1})) \quad \text{and} \quad A := \left\{\varphi \in \mathbb{R}^n : \ \varphi'(b - r\mathbf{1}) + z_\beta \left\|\sigma'\varphi\right\| = -\frac{C}{x}\right\}.$$

To show this, first note

$$(\varphi^*)'(b - r\mathbf{1}) + z_\beta \left\|\sigma'\varphi^*\right\| \overset{(3.3),(3.4)}{=} \lambda\Theta + |\lambda| z_\beta = \lambda(\Theta + z_\beta) = -\frac{C}{x}$$

implies $\varphi^* \in A$ and

$$(\varphi_*)'(b - r\mathbf{1}) + z_\beta \left\|\sigma'\varphi_*\right\| \overset{(3.3),(3.4)}{=} \mu\Theta + |\mu| z_\beta = \mu(\Theta - z_\beta) = -\frac{C}{x}$$

implies $\varphi_* \in A$. Next, if $\varphi \in A$, then

$$\begin{aligned} g(\varphi_*) \quad &= \quad x(1 + r + (\varphi_*)'(b - r\mathbf{1})) \overset{(3.3)}{=} x(1 + r + \mu\Theta) = x\left(1 + r - \frac{\frac{C}{x}\Theta}{\Theta - z_\beta}\right) \\ &\overset{(3.6)}{\leq} \quad x(1 + r + \varphi'(b - r\mathbf{1})) = g(\varphi) \overset{(3.7)}{\leq} x\left(1 + r - \frac{\frac{C}{x}\Theta}{\Theta + z_\beta}\right) \\ &= \quad x(1 + r + \lambda\Theta) \overset{(3.3)}{=} x(1 + r + (\varphi^*)'(b - r\mathbf{1})) = g(\varphi^*). \end{aligned}$$

This completes the proof. □

Theorem 5.3 (Closed-form solution to the discrete-time one-period mean-CaR optimization problem, part 2)**.** *Assume* (3.1), (3.5), *and*

$$\Theta + z_\beta > 0 \quad \textit{and} \quad C > 0.$$

The closed-form solution of problem (5.2) *is given by*

$$\varphi_* = \frac{\mu(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\Theta} \quad \textit{with} \quad \mu = -\frac{\frac{C}{x}}{\Theta - z_\beta}.$$

The expected wealth after one period is

$$\mathbb{E}(X^{\varphi_*}(1)) = x\left(1 + r + \mu\Theta\right)$$

with $\mathrm{CaR}(\varphi_*) = C$.

Proof. As in the proof of Theorem 5.2 and with the same g and A, it suffices to show that $\varphi_* \in A$ and

$$g(\varphi) \geq g(\varphi_*) = x(1 + r + \mu\Theta) \quad \text{for all} \quad \varphi \in A.$$

To show this, first note

$$(\varphi_*)'(b - r\mathbf{1}) + z_\beta \left\|\sigma'\varphi_*\right\| \overset{(3.3),(3.4)}{=} \mu\Theta + |\mu| z_\beta = \mu(\Theta - z_\beta) = -\frac{C}{x}$$

implies $\varphi_* \in A$. Next, if $\varphi \in A$, then

$$\begin{aligned} g(\varphi) \quad &= \quad x(1 + r + \varphi'(b - r\mathbf{1})) \overset{(3.6)}{\geq} x\left(1 + r - \frac{\frac{C}{x}\Theta}{\Theta - z_\beta}\right) \\ &= \quad x(1 + r + \mu\Theta) \overset{(3.3)}{=} x(1 + r + (\varphi_*)'(b - r\mathbf{1})) = g(\varphi_*). \end{aligned}$$

This completes the proof. □

Theorem 5.4 (Closed-form solution to the discrete-time one-period mean-CaR optimization problem, part 3)**.** *Assume* (3.1), (3.5), *and*

$$\Theta + z_\beta > 0 \quad \textit{and} \quad C < 0.$$

The closed-form solution of problem (5.2) *is given by*

$$\varphi_* = \frac{\lambda(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\Theta} \quad \textit{with} \quad \lambda = -\frac{\frac{C}{x}}{\Theta + z_\beta}.$$

The expected wealth after one period is

$$\mathbb{E}(X^{\varphi_*}(1)) = x\,(1 + r + \lambda\Theta)$$

with $\mathrm{CaR}(\varphi_*) = C$.

Proof. As in the proof of Theorem 5.2 and with the same g and A, it suffices to show that $\varphi_* \in A$ and

$$g(\varphi) \geq g(\varphi_*) = x(1 + r + \lambda\Theta) \quad \text{for all} \quad \varphi \in A.$$

To show this, first note

$$(\varphi_*)'(b - r\mathbf{1}) + z_\beta \left\|\sigma'\varphi_*\right\| \overset{(3.3),(3.4)}{=} \lambda\Theta + |\lambda| z_\beta = \lambda(\Theta + z_\beta) = -\frac{C}{x}$$

implies $\varphi_* \in A$. Next, if $\varphi \in A$, then

$$\begin{aligned} g(\varphi) &= x(1 + r + \varphi'(b - r\mathbf{1})) \overset{(3.7)}{\geq} x\left(1 + r - \frac{\frac{C}{x}\Theta}{\Theta + z_\beta}\right) \\ &= x(1 + r + \lambda\Theta) \overset{(3.3)}{=} x(1 + r + (\varphi_*)'(b - r\mathbf{1})) = g(\varphi_*). \end{aligned}$$

This completes the proof. □

Theorem 5.5 (Closed-form solution to the discrete-time one-period mean-CaR optimization problem, part 4)**.** *Assume* (3.1), (3.5), *and*

$$\Theta + z_\beta < 0 \quad \textit{and} \quad C < 0.$$

Then both (5.2) *and the mean-Capital-at-Risk problem* (5.1) *are unsolvable.*

Proof. Let A be the feasible set as in the proof of Theorem 5.2. If $\varphi \in A$, then

$$0 < -\frac{C}{x}\frac{\Theta}{\Theta - z_\beta} \overset{(3.6)}{\leq} \varphi'(b - r\mathbf{1}) \overset{(3.7)}{\leq} -\frac{C}{x}\frac{\Theta}{\Theta + z_\beta} < 0.$$

This contradiction shows $A = \emptyset$, and hence both (5.1) and (5.2) are unsolvable. □

Example 5.6. We calculate the maximal expected wealth with $\mathrm{CaR} = C$. Let

$$r = 0.05, \quad b = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.3 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}$$

and

$$x = 1000, \quad C = 20, \quad z_\beta = -1.64.$$

Then

$$\Theta + z_\beta = \left\| \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right\| - 1.64 \approx -0.203859$$

so that all assumptions of Theorem 5.2 are satisfied. Next,

$$\lambda = -\frac{\frac{20}{1000}}{\left\| \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right\| - 1.64} \approx 0.098107.$$

By Theorem 5.2, the optimal investment strategy is given by

$$\varphi^* = \frac{\lambda \cdot \begin{pmatrix} 0.01 & 0 & 0 \\ 0 & 0.09 & 0 \\ 0 & 0 & 0.04 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix}}{\left\| \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right\|} \approx \begin{pmatrix} 0.341564 \\ 0.113855 \\ 0.426955 \end{pmatrix}.$$

This means 34.1564% are invested in asset 1, 11.3855% are invested in asset 2, and 42.6955% are invested in asset 3. Now we calculate the expected wealth of this strategy:

$$\mathbb{E}(X^{\varphi}(1)) = 1000 \cdot \left(1 + 0.05 + (\varphi^*)' \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right) \approx 1190.895254.$$

We finally check if the CaR of this strategy really is 20:

$$\mathrm{CaR}(\varphi^*) = -1000 \cdot \left((-1.64) \cdot \lambda + (\varphi^*)' \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right) = 20.$$

6. One-Period Value-at-Risk Problem

In this section we introduce the discrete-time one-period mean-Value-at-Risk problem and provide a closed-form solution.

Definition 6.1 (Value-at-Risk)**.** $\mathrm{VaR}(\varphi) := \mu(\varphi)$.

We accept a certain amount as Value-at-Risk and we want to find the portfolio strategy which maximizes our expected wealth. We solve the optimization problem

$$\begin{cases} \max\limits_{\varphi \in \mathbb{R}^n} \mathbb{E}(X^{\varphi}(1)) \\ \text{s.t. } \mathrm{VaR}(\varphi) = C, \end{cases} \tag{6.1}$$

and we also solve the problem

$$\begin{cases} \min\limits_{\varphi \in \mathbb{R}^n} \mathbb{E}(X^{\varphi}(1)) \\ \text{s.t. } \mathrm{VaR}(\varphi) = C, \end{cases} \tag{6.2}$$

where C is the VaR at time $T = 1$. An overview of the results given in this section is displayed in Table 2.

Theorem 6.2 (Closed-form solution to the discrete-time one-period mean-VaR optimization problem, part 1)**.** *Assume* (3.1), (3.5), *and*

$$\Theta + z_\beta < 0 \quad \textit{and} \quad C < x(1 + r).$$

TABLE 2. Overview mean-Value-at-Risk problem

$\Theta + z_\beta$	$\frac{C}{x} - 1 - r$	Result	See
< 0	< 0	Found max and min	Theorem 6.2
> 0	< 0	Found min	Theorem 6.4
> 0	> 0	Found min	Theorem 6.3
< 0	> 0	$A = \emptyset$	Theorem 6.5

The closed-form solution of the one-period mean-Value-at-Risk problem (6.1) *is given by*

$$\varphi^* = \frac{\lambda(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\Theta} \quad \textit{with} \quad \lambda = \frac{\frac{C}{x} - 1 - r}{\Theta + z_\beta}.$$

The closed-form solution of problem (6.2) *is given by*

$$\varphi_* = \frac{\mu(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\Theta} \quad \textit{with} \quad \mu = \frac{\frac{C}{x} - 1 - r}{\Theta - z_\beta}.$$

The corresponding expected wealth after one period is

$$\mathbb{E}(X^{\varphi^*}(1)) = x\,(1 + r + \lambda\Theta) \quad \textit{and} \quad \mathbb{E}(X^{\varphi_*}(1)) = x\,(1 + r + \mu\Theta)\,,$$

respectively, with $\mathrm{VaR}(\varphi^*) = \mathrm{VaR}(\varphi_*) = C$*.*

Proof. Using (2.11) for $t = 1$ and (2.14), it suffices to show that $\varphi^*, \varphi_* \in A$ and

$$x(1 + r + \mu\Theta) = g(\varphi_*) \le g(\varphi) \le g(\varphi^*) = x(1 + r + \lambda\Theta) \quad \text{for all} \quad \varphi \in A,$$

where

$$g(\varphi) := x(1+r+\varphi'(b-r\mathbf{1})),\ A := \left\{\varphi \in \mathbb{R}^n :\ \varphi'(b - r\mathbf{1}) + z_\beta \|\sigma'\varphi\| = \frac{C}{x} - 1 - r\right\}.$$

To show this, first note

$$(\varphi^*)'(b - r\mathbf{1}) + z_\beta \|\sigma'\varphi^*\| \overset{(3.3),(3.4)}{=} \lambda\Theta + |\lambda| z_\beta = \lambda(\Theta + z_\beta) = \frac{C}{x} - 1 - r$$

implies $\varphi^* \in A$ and

$$(\varphi_*)'(b - r\mathbf{1}) + z_\beta \|\sigma'\varphi_*\| \overset{(3.3),(3.4)}{=} \mu\Theta + |\mu| z_\beta = \mu(\Theta - z_\beta) = \frac{C}{x} - 1 - r$$

implies $\varphi_* \in A$. Next, if $\varphi \in A$, then

$$\begin{aligned} g(\varphi_*) &= x(1 + r + (\varphi_*)'(b - r\mathbf{1})) \overset{(3.3)}{=} x(1 + r + \mu\Theta) \\ &= x\left(1 + r + \frac{\left(\frac{C}{x} - 1 - r\right)\Theta}{\Theta - z_\beta}\right) \overset{(3.6)}{\le} x(1 + r + \varphi'(b - r\mathbf{1})) \\ &= g(\varphi) \overset{(3.7)}{\le} x\left(1 + r + \frac{\left(\frac{C}{x} - 1 - r\right)\Theta}{\Theta + z_\beta}\right) \\ &= x(1 + r + \lambda\Theta) \overset{(3.3)}{=} x(1 + r + (\varphi^*)'(b - r\mathbf{1})) = g(\varphi^*). \end{aligned}$$

This completes the proof. $\square$

Theorem 6.3 (Closed-form solution to the discrete-time one-period mean-VaR optimization problem, part 2)**.** *Assume* (3.1), (3.5), *and*

$$\Theta + z_\beta > 0 \quad \text{and} \quad C < x(1+r).$$

The closed-form solution of problem (6.2) *is given by*

$$\varphi_* = \frac{\mu(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\Theta} \quad \text{with} \quad \mu = \frac{\frac{C}{x} - 1 - r}{\Theta - z_\beta}.$$

The expected wealth after one period is

$$\mathbb{E}(X^{\varphi_*}(1)) = x\,(1 + r + \mu\Theta)$$

with $\mathrm{VaR}(\varphi_*) = C$*.*

Proof. As in the proof of Theorem 6.2 and with the same g and A, it suffices to show that $\varphi_* \in A$ and

$$g(\varphi) \geq g(\varphi_*) = x(1 + r + \mu\Theta) \quad \text{for all} \quad \varphi \in A.$$

To show this, first note

$$(\varphi_*)'(b - r\mathbf{1}) + z_\beta \, \|\sigma'\varphi_*\| \overset{(3.3),(3.4)}{=} \mu\Theta + |\mu| z_\beta = \mu(\Theta - z_\beta) = \frac{C}{x} - 1 - r$$

implies $\varphi_* \in A$. Next, if $\varphi \in A$, then

$$\begin{aligned} g(\varphi) &= x(1 + r + \varphi'(b - r\mathbf{1})) \overset{(3.6)}{\geq} x\left(1 + r + \frac{\left(\frac{C}{x} - 1 - r\right)\Theta}{\Theta - z_\beta}\right) \\ &= x(1 + r + \mu\Theta) \overset{(3.3)}{=} x(1 + r + (\varphi_*)'(b - r\mathbf{1})) = g(\varphi_*). \end{aligned}$$

This completes the proof. □

Theorem 6.4 (Closed-form solution to the discrete-time one-period mean-VaR optimization problem, part 3)**.** *Assume* (3.1), (3.5), *and*

$$\Theta + z_\beta > 0 \quad \text{and} \quad C > x(1+r).$$

The closed-form solution of problem (6.2) *is given by*

$$\varphi_* = \frac{\lambda(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\Theta} \quad \text{with} \quad \lambda = \frac{\frac{C}{x} - 1 - r}{\Theta + z_\beta}.$$

The expected wealth after one period is

$$\mathbb{E}(X^{\varphi_*}(1)) = x\,(1 + r + \lambda\Theta)$$

with $\mathrm{VaR}(\varphi_*) = C$*.*

Proof. As in the proof of Theorem 6.2 and with the same g and A, it suffices to show that $\varphi_* \in A$ and

$$g(\varphi) \geq g(\varphi_*) = x(1 + r + \lambda\Theta) \quad \text{for all} \quad \varphi \in A.$$

To show this, first note

$$(\varphi_*)'(b - r\mathbf{1}) + z_\beta \, \|\sigma'\varphi_*\| \overset{(3.3),(3.4)}{=} \lambda\Theta + |\lambda| z_\beta = \lambda(\Theta + z_\beta) = \frac{C}{x} - 1 - r$$

implies $\varphi_* \in A$. Next, if $\varphi \in A$, then

$$\begin{aligned} g(\varphi) &= x(1+r+\varphi'(b-r\mathbf{1})) \overset{(3.7)}{\geq} x\left(1+r+\frac{\left(\frac{C}{x}-1-r\right)\Theta}{\Theta+z_\beta}\right) \\ &= x(1+r+\lambda\Theta) \overset{(3.3)}{=} x(1+r+(\varphi_*)'(b-r\mathbf{1})) = g(\varphi_*). \end{aligned}$$

This completes the proof. □

Theorem 6.5 (Closed-form solution to the discrete-time one-period mean-VaR optimization problem, part 4)**.** *Assume* (3.1), (3.5), *and*

$$\Theta + z_\beta < 0 \quad \textit{and} \quad C > x(1+r).$$

Then both (6.2) *and the mean-Value-at-Risk problem* (6.1) *are unsolvable.*

Proof. Let A be the feasible set as in the proof of Theorem 6.2. If $\varphi \in A$, then

$$0 < \left(\frac{C}{x}-1-r\right)\frac{\Theta}{\Theta - z_\beta} \overset{(3.6)}{\leq} \varphi'(b-r\mathbf{1}) \overset{(3.7)}{\leq} \left(\frac{C}{x}-1-r\right)\frac{\Theta}{\Theta+z_\beta} < 0.$$

This contradiction shows $A = \emptyset$, and hence both (6.1) and (6.2) are unsolvable. □

Example 6.6. We calculate the maximal expected wealth with $\mathrm{VaR} = C$. Let r, b, σ, x, and z_β be as in Example 5.6 and let $C = 1030$. Thus $C < x(1+r)$ so that all assumptions of Theorem 6.2 are satisfied. Next,

$$\lambda = \frac{\frac{1030}{1000} - 1 - 0.05}{\left\| \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right\| - 1.64} \approx 0.098107.$$

By Theorem 6.2, the optimal investment strategy is given by

$$\varphi^* = \frac{\lambda \cdot \begin{pmatrix} 0.01 & 0 & 0 \\ 0 & 0.09 & 0 \\ 0 & 0 & 0.04 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix}}{\left\| \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right\|} \approx \begin{pmatrix} 0.341564 \\ 0.113855 \\ 0.426955 \end{pmatrix}.$$

This means 34.1564% are invested in asset 1, 11.3855% are invested in asset 2, and 42.6955% are invested in asset 3. Now we calculate the expected wealth of this strategy:

$$\mathbb{E}(X^\varphi(1)) = 1000 \cdot \left(1 + 0.05 + (\varphi^*)' \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix}\right) \approx 1190.895254.$$

We finally check if the VaR of this strategy really is 1030:

$$\mathrm{VaR}(\varphi^*) = 1000 \cdot \left((-1.64)\cdot\lambda + 1 + 0.05 + (\varphi^*)' \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix}\right) = 1030.$$

7. Multi-Period Mean-Variance Problem

In this section we introduce the multi-period mean-variance problem (see also [7]) and provide a closed-form solution. We solve the optimization problem

$$\left\{\begin{array}{l}\min\limits_{\varphi\in\mathbb{R}^n} \mathbb{V}\mathrm{ar}(X^\varphi(T))\\ \text{s.t. } \mathbb{E}(X^\varphi(T)) \geq C,\end{array}\right. \tag{7.1}$$

where C is the expected terminal wealth at time T. We assume that the expected wealth of the investor is greater than the wealth of the risk-free asset.

Theorem 7.1 (Closed-form solution of the discrete-time multi-period mean-variance optimization problem)**.** *Assume* (3.1) *and* (3.5)*. The closed-form solution of the multi-period mean-variance problem* (7.1) *is given by*

$$\varphi_* = \frac{\lambda(\sigma\sigma')^{-1}(b-r\mathbf{1})}{\Theta} \quad \textit{with} \quad \lambda = \frac{\sqrt[T]{\frac{C}{x}}-1-r}{\Theta}.$$

The expected wealth after T periods is C with variance

$$\mathbb{V}\mathrm{ar}(X^\varphi(T)) = x^2\left[\left(\left(\frac{C}{x}\right)^{\frac{2}{T}}+\lambda^2\right)^T - \left(\frac{C}{x}\right)^2\right].$$

Proof. Using (2.11) and (2.12) for $t = T$, it suffices to show that $\varphi_* \in A$ and

$$g(\varphi) \geq g(\varphi_*) = x^2\left[\left(\left(\frac{C}{x}\right)^{\frac{2}{T}}+\lambda^2\right)^T - \left(\frac{C}{x}\right)^2\right] \quad \text{for all} \quad \varphi\in A,$$

where

$$g(\varphi) := x^2\left[\left((1+r+\varphi'(b-r\mathbf{1}))^2+\varphi'\sigma\sigma'\varphi\right)^T - (1+r+\varphi'(b-r\mathbf{1}))^{2T}\right]$$

and

$$A := \left\{\varphi\in\mathbb{R}^n : \ 1+r+\varphi'(b-r\mathbf{1}) \geq \sqrt[T]{\frac{C}{x}}\right\}.$$

To show this, first note that

$$\begin{aligned} x\left(1+r+(\varphi_*)'(b-r\mathbf{1})\right)^T &\overset{(3.3)}{=} x(1+r+\lambda\Theta)^T \\ &= x\left(1+r+\frac{\sqrt[T]{\frac{C}{x}}-1-r}{\Theta}\Theta\right)^T = C \end{aligned}$$

implies $\varphi_* \in A$. Next, if $\varphi\in A$, then

$$\|\sigma'\varphi\|^2 \overset{(3.2)}{\geq} \frac{|\varphi'(b-r\mathbf{1})|^2}{\Theta^2} \geq \left|\frac{\varphi'(b-r\mathbf{1})}{\Theta}\right|^2 \geq \lambda^2 \tag{7.2}$$

and thus

$$
\begin{aligned}
g(\varphi) &= x^2\left[\left((1+r+\varphi'(b-r\mathbf{1}))^2+\varphi'\sigma\sigma'\varphi\right)^T-(1+r+\varphi'(b-r\mathbf{1}))^{2T}\right]\\
&\overset{(7.2)}{\geq} x^2\left[\left((1+r+\varphi'(b-r\mathbf{1}))^2+\lambda^2\right)^T-(1+r+\varphi'(b-r\mathbf{1}))^{2T}\right]\\
&\overset{(\varphi\in A)}{\geq} x^2\left[\left(\left(1+r+\sqrt[T]{\frac{C}{x}}-1-r\right)^2+\lambda^2\right)^T-\left(1+r+\sqrt[T]{\frac{C}{x}}-1-r\right)^{2T}\right]\\
&= x^2\left[\left(\left(\frac{C}{x}\right)^{\frac{2}{T}}+\lambda^2\right)^T-\left(\frac{C}{x}\right)^2\right]\\
&= x^2\left[\left((1+r+\lambda\Theta)^2+\lambda^2\right)^T-(1+r+\lambda\Theta)^{2T}\right]\\
&\overset{(3.3)}{\underset{(3.4)}{=}} x^2\left[\left((1+r+(\varphi_*)'(b-r\mathbf{1}))^2+(\varphi_*)'\sigma\sigma'\varphi_*\right)^T-(1+r+(\varphi_*)'(b-r\mathbf{1}))^{2T}\right]\\
&= g(\varphi_*),
\end{aligned}
$$

where in the second inequality sign we have used Lemma 3.3. □

As an immediate consequence of Theorem 7.1 we get that the mean-variance is a function of the expected terminal wealth. An investor is now able to plot a graph for different expected terminal wealths. Let us denote with $\omega := \mathbb{E}(X^\varphi(T))$ the expected wealth after T periods. Now we can plug it into the result of Theorem 7.1 to get

$$
\mathbb{V}\mathrm{ar}(\omega) = x^2\left[\left(\left(\frac{\omega}{x}\right)^{\frac{2}{T}}+\left(\frac{\sqrt[T]{\frac{\omega}{x}}-1-r}{\Theta}\right)^2\right)^T-\left(\frac{\omega}{x}\right)^2\right].
$$

If we know our desirable expected terminal wealth, then we can calculate λ and the portfolio construction strategy. Another way is that we accept a certain amount as variance, and then we calculate ω and set this equal to C. Then we are able to calculate the optimal portfolio.

Example 7.2. Let r, b, σ, and x be as in Example 4.3 and let $C = 1110$ and $T = 2$. Now we calculate

$$
\lambda = \frac{\sqrt{\frac{1110}{1000}}-1-0.05}{\left\|\begin{pmatrix}0.2 & 0.01 & 0.03\\ 0.1 & 0.3 & 0.04\\ 0.05 & 0.03 & 0.1\end{pmatrix}^{-1}\begin{pmatrix}0.05\\ 0.15\\ 0.25\end{pmatrix}\right\|} \approx 0.001416.
$$

Then we find the variance of our portfolio with expected terminal wealth C as

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}(\varphi_*) &= x^2\left[\left(\left(\frac{C}{x}\right)^{\frac{2}{T}}+\lambda^2\right)^T-\left(\frac{C}{x}\right)^2\right]\\
&= 1000^2\left[\left(\frac{1110}{1000}+\lambda^2\right)^2-\left(\frac{1110}{1000}\right)^2\right] \approx 4.453.
\end{aligned}
$$

This is the minimal variance for the portfolio with an expected terminal wealth of 1110 at time 2. By Theorem 7.1, the optimal investment strategy is given by

$$\varphi_* = \frac{\lambda \cdot \begin{pmatrix} 0.041 & 0.0242 & 0.0133 \\ 0.0242 & 0.1016 & 0.018 \\ 0.0133 & 0.018 & 0.0134 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix}}{\left\| \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right\|} \approx \begin{pmatrix} -0.003773 \\ -0.001042 \\ 0.015641 \end{pmatrix}.$$

This means we invest 1.5641% of our initial wealth in asset 3. The rest is invested risk free. Now we check if the expected wealth at time 2 really is 1110:

$$\mathbb{E}(X^{\varphi}(2)) = 1000 \cdot \left(1 + 0.05 + (\varphi_*)' \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right)^2 = 1110.$$

Remark 7.3. The presented results can also be generalized from difference equations to dynamic equations on isolated time scales (see [2]). This will be done in a forthcoming paper of the authors.

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